Bilkent University
Department of Mathematics

PROBLEM OF THE MONTH

June 2005

Problem: Prove that for arbitrary triangle $\triangle ABC$

$$|AB|\cos(B\widehat{C}A)+|BC|\cos(B\widehat{A}C)+|AC|\cos(A\widehat{B}C)\leq\frac{|AB|+|BC|+|AC|}{2}.$$

Solution: Let $|AB|=c, |BC|=a, |AC|=b$, $\widehat{BAC}=\widehat{A}$, $\widehat{ABC}=\widehat{B}$, and $\widehat{BCA}=\widehat{C}$. Let us prove that $a\cos\widehat{A}+b\cos\widehat{B}\leq c$. Indeed, since $c=a\cos\widehat{B}+b\cos\widehat{A}$, our inequality is equivalent to the inequality

$$(a-b)(\cos\widehat{A}-\cos\widehat{B})\leq 0.$$

The last inequality is correct, since $\cos x$ is a decreasing function on the interval $[0,\pi]$ and $\cos\widehat{A}\geq\cos\widehat{B}$ if $a\geq b$. Similarly we prove that

$$a\cos\widehat{A}+c\cos\widehat{C}\leq b \quad\text{and}\quad b\cos\widehat{B}+c\cos\widehat{C}\leq a.$$

The sum of these inequalities gives the desired inequality.