Bilkent University
Department of Mathematics

Problem Of The Month

April 2005

Problem: Suppose that, for all $-1 < x < 1$, the following inequality

$$ax^2 + bx + c \leq \frac{1}{\sqrt{1-x^2}}$$

is held. Find the maximum possible value of

$$\frac{a}{2} + c.$$

Solution: Put $x = \pm 1/\sqrt{2}$ into the inequality:

$$\frac{a}{2} + \frac{b}{\sqrt{2}} + c \leq \sqrt{2}$$

$$\frac{a}{2} - \frac{b}{\sqrt{2}} + c \leq \sqrt{2}$$

The sum of these inequalities gives

$$\frac{a}{2} + c \leq \sqrt{2}$$

Let us show that $\frac{a}{2} + c$ can take $\sqrt{2}$. Indeed, if $a = \sqrt{2}$, $b = 0$, $c = \frac{\sqrt{2}}{2}$ then our inequality takes the following form:

$$\sqrt{2}x^2 + \frac{\sqrt{2}}{2} \leq \frac{1}{\sqrt{1-x^2}}$$

The last inequality is a consequence of the arithmetic-geometric inequality:

$$\left(\sqrt{2}x + \frac{\sqrt{2}}{2}\right) \cdot \sqrt{1-x^2} = \sqrt{(x^2 + \frac{1}{2})(x^2 + \frac{1}{2})(2 - 2x^2)}$$

$$\leq \sqrt{\left(\frac{x^2 + \frac{1}{2} + x^2 + \frac{1}{2} + 2 - 2x^2}{3}\right)^3} = 1.$$

Thus, the maximum of $\frac{a}{2} + c$ is $\sqrt{2}$.