Bilkent University
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Problem Of The Month

April 2026

Problem:

Alice and Bob play a game on a grid consisting of 2026 rows and 9 columns. At the beginning of the game Alice inserts a number to each unit square such that all numbers of each row are different. After that they take turns to make move starting with Bob. Bob in each move chooses a number not chosen before and colour all unit squares containing this number to red. Alice in each move chooses a number not chosen before and colour all unit squares containing this number to blue. Suppose that Alice can guarantee that after colouring all unit squares there are at least N rows containing only blue unit squares. Determine the maximal possible value of N.

Solution:

Answer: 3.

Let us show that Alice can guarantee that after colouring of all unit squares of a grid there are at least 3 rows containing only blue unit squares. Let us numerate columns of the grid from the left to the right by $1, 2, \ldots, 9$. Alice divides the first column to consecutive blocks $B(1,1)$ and $B(1,2)$ of sizes 1013 and 1013 starting from the top. These blocks consist of unit squares of first 1013 and the next 1013 rows, consecutively. Similarly starting from the top Alice divides the second column to blocks $B(1,1,1), B(1,1,2), B(1,2,1), B(1,2,2)$ of sizes $[1013/2], [1013/2]+1, [1013/2], [1013/2]+1$. Alice similarly divides the third column to blocks $B(1,1,1,1), B(1,1,1,2), B(1,1,2,1)$, $B(1,1,2,2), B(1,2,1,1)$, $B(1,2,1,2)$, $B(1,2,2,1), B(1,2,2,2)$ of sizes $506, 507, 506, 507, 506$, $507, 506, 507$. Similarly Alice divides the fourth column to 16 blocks, fifth to 32 blocks, sixth to 64 blocks, seventh to 128 blocks, eights to 256 blocks. Finally Alice divides the ninth column to 512 blocks of sizes 3 and 4. Alice inserts the same number to all unit squares of each block and different numbers to different blocks. We say that the numbers inserted to blocks $B(1,1)$ and $B(1,2)$ are conjugated. Similarly all pairs of numbers inserted to pairs of blocks like $B(a,b,c,...,d,1)$ and $B(a,b,c,...,d,2)$ are conjugated. It can be readily shown that Alice in each move by choosing a number conjugated to the number chosen by Bob in his last move can guarantee the existence of 3 rows containing only blue unit squares.

Now let us show that Bob can guarantee that there are at most 3 rows containing only blue unit squares. Let the weight of each row containing k blue unit squares and no red square is 2^k and the weight of any row containing at least one red square is 0. For each number t not chosen before let $W(t)$ be the total weight of all rows containing the number t. In each move Bob chooses number t' for which $W(t')$ is maximal and colours all unit squares containing t' to red. Suppose that after that Alice chooses a number s. After Alice's move the weight of each row containing s will be doubled. Since $W(t') \geq W(s)$ after these two moves the total weight of all rows can not increase. The total weight of all rows at the very beginning is 2026. The weight of each totally coloured row containing only blue unit squares is 2^9. Hence if Bob follows his strategy at the end the total number of such blue rows will be at most $\lfloor \frac{2026}{2^9} \rfloor = 3$.