Bilkent University
Department of Mathematics

Problem Of The Month

March 2026

Problem:

On each vertex of a regular $n \geq 3$ sided polygon there is a real number so that the sum of these n numbers is zero. For any two vertices of the polygon with numbers x and y consider the line passing through these vertices dividing the polygon into two parts. Let the sum of numbers written on one of these parts be A and the sum of numbers written on the other part be B. If

$$|x - y| \geq |A - B|$$

we say that this pair of vertices is *good*. For each fixed n find the minimal possible number of good vertex pairs.

Solution:

Answer: $n - 1$.

If one of the numbers is equal to $n - 1$ and all the others are equal to -1 then the total number of good pairs is $n - 1$.

Let us show that the total number of good pairs is not less than $n - 1$. Let the vertices of the polygon are $v_1, v_2, \ldots, v_n$, where $v_{i+n} = v_i$. Suppose that the vertices of the polygon lying e on some some circle. For each $k = 1, 2, \ldots, n - 1$ consider the arcs $L(v_i, v_{i+k-1})$ containing k consecutive vertices $v_i, v_{i+1}, \ldots, v_{i+k-1}$ and let $S(L(v_i, v_{i+k-1}))$ be the sum of all numbers written on $L(v_i, v_{i+k-1})$. Suppose that $S(L(v_i, v_{i+k-1})) \geq 0$ and $S(L(v_{i+1}, v_{i+k})) \leq 0$ for some index i. Then we show that the pair (v_i, v_{i+k}) is good. Let x_i and x_{i+k} be the numbers written on the vertices v_i and v_{i+k}, respectively. Let A be the sum of numbers written on the vertices $v_{i+1}, v_{i+2}, \ldots, v_{i+k-1}$ and B be the sum of the numbers written on the vertices $v_{i+k+1}, v_{i+k+2}, \ldots, v_{i-1}$. Then $x_i + A \geq 0 \geq A + x_{i+k}$. Since by problem conditions $x_i + A + x_{i+k} + B = 0$ we also get that $B + x_i \geq 0 \geq x_{i+k} + B$. Therefore, $x_i - x_{i+k} \geq A - B$ and $x_i - x_{i+k} \geq B - A$ and hence (v_i, v_{i+k}) is good. Since the sum of all numbers is equal to zero there is also an index j for which $S(L(v_j, v_{j+k-1})) \leq 0$ and $S(L(v_{j+1}, v_{j+k})) \geq 0$ and similarly the pair (v_j, v_{j+k}) is good. Since each good pair is counted twice we see that for each $k = 1, 2, \ldots, n - 1$ there is at least one good pair and we are done.