Bilkent University
Department of Mathematics

Problem Of The Month

February 2026

Problem:

Let $0 \leq x_1, x_2, \ldots, x_{2026} \leq 1$ be real numbers. Find the minimal possible value of

$$\sum_{i=1}^{2026} x_i x_{i+1} + \sum_{i=1}^{2026} x_i x_{i+2} + \sum_{i=1}^{2026} x_i x_{i+3} - \sum_{i=1}^{2026} x_i$$

where $x_{2027} = x_1$, $x_{2028} = x_2$ and $x_{2029} = x_3$.

Solution:

Answer: -506.

For each index i the sum of all terms including x_i is

$$x_i(x_{i-3} + x_{i-2} + x_{i-1} - 1 + x_{i+1} + x_{i+2} + x_{i+3}).$$

Suppose that one of the points where the minimum is achieved is $(x'_1, x'_2, \ldots, x'_{2026})$. Therefore if $x'_{i-3} + x'_{i-2} + x'_{i-1} + x'_{i+1} + x'_{i+2} + x'_{i+3} - 1 > 0$ for some index i then $x'_i = 0$ and if $x'_{i-3} + x'_{i-2} + x'_{i-1} + x'_{i+1} + x'_{i+2} + x'_{i+3} - 1 < 0$ for some index i then $x'_i = 1$. If $x'_{i-3} + x'_{i-2} + x'_{i-1} + x'_{i+1} + x'_{i+2} + x'_{i+3} - 1 = 0$ then let us put $x'_i = 0$. Hence the minimal value will be achieved for $x'_i \in \{0, 1\}$. Thus, if $x'_i = 1$ then $x'_{i-3} = x'_{i-2} = x'_{i-1} = x'_{i+1} = x'_{i+2} = x'_{i+2} = 0$. Therefore, the minimal value can not be less than $-[\frac{2026}{4}] = -506$.

The value -506 will be achieved at $(x'_1, x'_2, \ldots, x'_{2026}) = (0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, \ldots, 1, 0, 0, 0,)$.