Bilkent University
Department of Mathematics

Problem Of The Month

December 2025

Problem:

For each positive integer n having at least 6 positive divisors let $d_6(n)$ be the sixth smallest positive divisor of n. For example $d_6(5040) = 6$. Find all positive integers n such that

$$d_6 = \frac{3 + \sqrt{4n+9}}{2}.$$

Solution:

Answer: $n = 208$ and $n = 304$.

Note that $d_6 \geq 6$. Since $d_6 - 3 = \dfrac{-3 + \sqrt{4n+9}}{2}$ we get $d_6(d_6 - 3) = n$. Let us show that neither $d_6 - 1$ nor $d_6 - 2$ is a divisor of n. Note that $d_6 - 1 \mid (d_6 - 1)^2 - (d_6 - 1) = d_6^2 - 3d_6 + 2 = n + 2$. Then if $d_6 - 1 \mid n$ then $d_6 - 1 \mid 2$, contradicting with $d_6 \geq 6$. Similarly note that $d_6 - 2 \mid (d_6 - 2)^2 + (d_6 - 2) = d_6^2 - 3d_6 + 2 = n + 2$. Then if $d_6 - 2 \mid n$ then $d_6 - 1 \mid 2$, contradicting with $d_6 \geq 6$. Therefore, the fifth smallest divisor of n is $d_5 = d_6 - 3$. Since $d_6(d_6 - 3) = n$ we get that n has exactly 10 positive divisors.

Since one of d_6 and $d_6 - 3$ is even n is also even. Therefore, $n = 2^9$, $n = 2p^4$ or $n = 2^4 p$, where p is an odd prime.

$n = 2^9$ does not satisfy problem conditions. If $n = 2p^4$ then $d_5 = p^2$ and $d_6 = 2p^2$ and we get a contradiction $3 = d_6 - d_5 = p^2$.

Let $n = 2^4 p$. $p = 3$ does not satisfy problem conditions: $p > 3$. Since $d_6(d_6 - 3) = 16p$ and the divisors $d_6 > 1$ and $d_6 - 3 > 1$ are relatively prime one of these divisors is equal to 16. Thus, solutions are $208 = 2^4 \cdot 13$ and $304 = 2^4 \cdot 19$.