Bilkent University
Department of Mathematics

Problem Of The Month

November 2025

Problem:

Let $P_1, \ldots, P_{1024}$ be distinct points marked on a circle, and let $a_1, \ldots, a_{1024}$ be distinct real numbers written on these points, respectively. For each point Q on this circle different than $P_1, \ldots, P_{1024}$, we say that P_i is Q-*good* if a_i is the greatest number on at least one of the two arcs P_iQ on this circle. Let the score of Q be the number of $Q-$good points on the circle. Determine the greatest integer k such that regardless of the values of $a_1, \ldots, a_{1024}$, there exists a point Q with score at least k.

Solution:

First, we will prove that for any configuration, there is a point Q with score at least 11. Without loss of generality, assume that $a_1 > a_2 > \cdots > a_{1024}$. It is clear that, for any point Q, P_1 and P_2 are Q-good points. Consider the arc P_1P_2 (excluding the endpoints) which contains at least 511 points, and let a_{i_3} be the largest number on this arc. Then, for any point Q on this arc, P_{i_3} is a Q-good point as well. Then, choose the arc that is inside the previous arc and contains at least 255 points among $P_{i_3}P_1$ and $P_{i_3}P_2$ (excluding the endpoints). Let a_{i_4} be the largest point on this arc, then P_{i_4} is a Q-good for any point Q on this arc. Thus, we can continue this process by letting $i_1 = 1$, $i_2 = 2$ and for each $k \geq 3$, choosing i_k such that a_{i_k} is the greatest number on the arc $P_{i_{k-1}}P_{i_{k-2}}$ or $P_{i_{k-1}}P_{i_{k-3}}$ which lies inside the previously chosen arc and is containing at least $2^{12-k} - 1$ points (excluding the endpoints). So, this process will continue as long as we can choose the next point, which is possible for at least 11 steps. So, on the remaining arc, we will have a point Q such that P_{i_k} is Q-good for $k = 1, 2, \ldots, 11$.

Now let us construct an example where no point has a score greater than 11. Consider a regular 1024-gon with vertices on the circle. We will distribute the numbers $1, 2, \ldots, 1024$ to the vertices of this polygon. At the first step we write 1023 and 1024 to two points forming a diameter of this circle. At the second step we write 1022 and 1021 to the midpoints of the newly arised two arcs. At the third step we write 1020, 1019, 1018 and 1017 to the midpoints of the newly arised four arcs at the second step. Consecutively, for $k = 4, 5, \ldots, 10$ at k-st step we write 2^{k-1} largest numbers out of remaining numbers to the midpoints of the 2^{k-1} arcs arised at the $(k-1)$-st step. It is easy to verify that, in each step the score on the circle for any point increases by 1. Hence, after writing all the numbers, we will end up with a configuration where every point has a score 11. We are done.