



Bilkent University
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PROBLEM OF THE MONTH

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Problem:

The function $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}_{\geq 0}$ is defined by $f(1) = 0$ and $f(n) = \alpha_1 + \dots + \alpha_k$, where $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ is the prime factorization of $n > 1$. If for some infinite arithmetic progression $\{x_i\}_{i=1}^{\infty}$ and for some integer $a > 1$ the sequence $\{f(a^{x_i} - 1)\}_{i=1}^{\infty}$ is also an arithmetic progression then we say that the sequence $\{x_i\}_{i=1}^{\infty}$ is *good*. Find all good sequences.

Solution:

Any constant sequence satisfies problem condition. Now assume that there exists non-constant arithmetic sequence $x_n = a + bn$ satisfying problem conditions and let $d = (a, b)$. For any $m, n \in \mathbb{Z}^+$, it can be readily shown that $\gamma(mn) = \gamma(m) + \gamma(n)$. Thus, letting $y_i = \frac{a^{x_i} - 1}{a^d - 1}$, we have $\gamma(y_i) = \gamma(a^{x_i} - 1) - \gamma(a^d - 1)$ which implies that $\{\gamma(y_i)\}_{i=1}^{\infty}$ is also an arithmetic sequence. By Dirichlet's Theorem, there exist infinitely many terms n such that $\frac{a + bn}{d} = p$ is a prime number.

It is a well known fact that any prime divisor of the number $\frac{t^p - 1}{t - 1}$ is either p or congruent to 1 modulo p , thus at least p . Letting $t = a^d$, we get that for arbitrarily large primes p , there exists a value n such that all prime divisors of y_n are greater than p . On the other hand, $\gamma(a^{x_i}) = \gamma(a)x_i$ is an arithmetic sequence as well. Let q be the greatest prime divisor of a , and N be an arbitrary positive integer. Taking any prime $p > q^N$, we get that there exists an arbitrarily large positive integer n such that $\frac{\gamma(a^{x_n})}{\gamma(y_n)} > N$, but this is not correct for two non-constant arithmetic sequences. Thus, at least one the sequences must be constant.

The sequence $\{y_i\}_{i=1}^{\infty}$ cannot be constant, since for any $x_i = a + bi$, we know that $(b+1)x_i$ also appears in this sequence as well, and we have

$$\frac{a^{x_i} - 1}{a^d - 1} \mid \frac{a^{(b+1)x_i} - 1}{a^d - 1}$$

where the second term is strictly larger than the first term if $b \neq 0$. This implies that, for $x_j = (b+1)x_i$ we have $y_j > y_i$, therefore Y cannot be a constant sequence. Thus, the sequence X must be constant. We are done.