Bilkent University
Department of Mathematics

Problem Of The Month

September 2025

Problem:

Find all pairs (p, n), where p is a prime number and $n < p$ is a positive integer number such that $pn(p+1)(n+1)+1$ is a perfect square.

Solution: Answer: $(p, n) = (p, p-2)$, where $p > 2$.

Let $pn(p+1)(n+1)+1 = m^2$. Readily $n \neq p$ and $m > np$. Let $m = np + k$. Then

$$pn(p+1)(n+1)+1 = (pn+k)^2 = pn(pn+2k)+k^2$$

By simplifying we get

$$pn(p+n+1-2k) = k^2 - 1 \qquad (\dagger)$$

$k \neq 1$ since otherwise $p + n = 1$. Hence $k > 1$ and $2k < p + n + 1 \leq 2p$ and hence $k < p$. Then since $p | k^2 - 1 = (k-1)(k+1)$ we get $p = k + 1$. Inserting it to ($\dagger$) we get $n(n - p + 3) = p - 2$. This quadratic equation with respect to n has two solutions: $n = -1$ and $n = p - 2$. Thus, the only possibility is $n = p - 2$ which satisfies problem conditions: if $n = p - 2$ then

$$pn(p+1)(n+1)+1 = (p^2 - (p+1)^2)^2.$$