Bilkent University
Department of Mathematics

Problem Of The Month

July-August 2025

Problem:

Let a, b, c be pairwise relatively prime positive integers satisfying $a > bc$. For any two positive integers $m < n$ we say that m is a *successor* of n, if for every pile of stones with weights a, b, c and total weight n, it is possible to remove some stones and obtain a new pile with total weight m. Find the greatest positive integer that does not have any successor.

Solution: Answer: abc.

We first show that abc has no successor. Suppose that it has a successor m. Consider a pile consisting of bc stones each having weight a. Thus, m has to be a multiple of a. Similarly, m is divisible by b and c as well. As those three numbers are relatively prime, m must be a multiple of abc which contradicts with $m < abc$.

We next show that every $n > abc$ has a successor. Let $n = aq + r$ with $0 \leq r < a$. By the Chinese Remainder theorem the system

$$\begin{aligned} m &\equiv r \pmod{a} \\ m &\equiv 0 \pmod{b} \\ m &\equiv 0 \pmod{c} \end{aligned}$$

has a solution $m \in \{1, 2, \ldots, abc\}$. Let $m = ak + r = bcl$. Consider a pile consisting of x stones of weight a, y stones of weight b and z stones of weight c such that $n = ax + by + cz$. Then there exists a non-negative integer s such that $by + cz = as + r$.

Case 1: $s \leq k$. Then we have $x + s = q$ and $s \leq k \leq bc \leq q = x + s$. Removing $x - k + s$ stones of weight a gives a pile with total weight m.

Case 2: $s > k$. Let $y = cy_1 + y_2$, $0 \leq y_2 < c$ and $c = bz_1 + z_2$, $0 \leq z_2 < b$. Then, as

$$by + cz = bc(y_1 + z_1) + by_2 + cz_2 < bc(y_1 + z_1 + 2)$$

and

$$bc(l+1) = bcl + bc < ak + r + a \leq as + r = by + cz,$$

we obtain $l + 1 < y_1 + z_1 + 2$. Thus, $l \leq y_1 + z_1$. Therefore, leaving a pile consisting of cy_1 stones of weight b and $b(l - y_1)$ stones of weight c gives a pile with total weight m, so we are done.