**Bilkent University**
**Department of Mathematics**

# Problem Of The Month

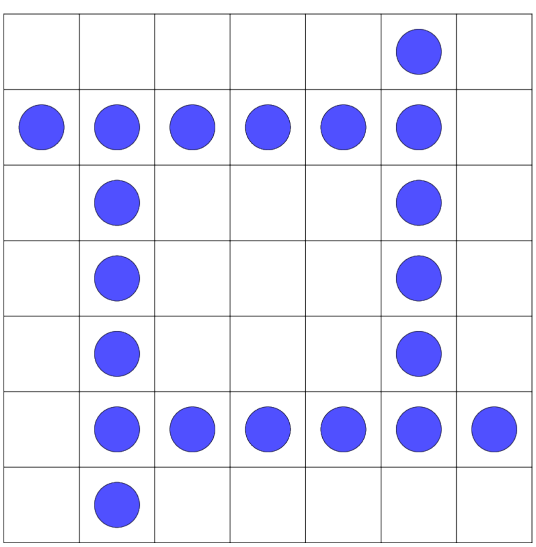
June 2025

**Problem:**

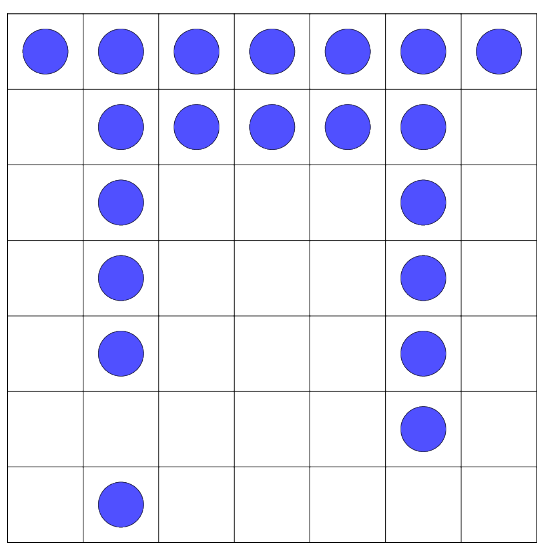
Let $n \geq 4$ is an integer. Suppose that some unit squares of $n \times n$ grid are marked. We say that two distinct marked unit squares are *line related* if they are on the same line and there is no any other marked unit square on the same line located between them. We say that two distinct marked unit squares are *column related* if they are on the same column and there is no any other marked unit square on the same column located between them. We say that two marked unit squares are *related* if they are line or column related. Find the maximal possible number of marked unit squares on the $n \times n$ grid if any marked unit square is related with an odd number of marked unit squares.

**Solution:** Answer: $4n - 8$.

Examples (for n=7):



or



Let us show that the number of marked unit squares on the chessboard satisfying conditions can not be greater than $4n-8$.

We define four directions: up, down, right and left. For each marked square we draw rays starting at the center of the unit square directed to the direction in which there is no any marked square. Each of these rays intersects a unit side of the length 1 located on the boundary of the grid. By definitions any unit boundary side is intersected by at most one ray. A marked square which is related to only one marked square is called *one*-marked, a marked square which is related to three marked squares is called *three*-marked. To each one-marked square we assign three unit sides intersected by rays starting at this square. To each three-rook we assign the unit side intersected by the ray starting at this square.

Lemma 1. If there are at least 4 one-marked squares then the total number of marked squares is at most $4n-8$.

Proof: Suppose that there are $a$ one-marked and $b$ three-marked squares. The number of unit sides assigned to each one-marked square is 3 and the number of unit sides assigned to each three-marked square is 1. Since the total length of the grid boundary $4n$ we get $3a+b\leq 4n$. Since $a\geq 4$ we get $a+b\leq 4n-2a\leq 4n-8$. Done.

Lemma 2. Suppose that there are at least $4n-8$ marked squares on the grid. Then the number one-marked squares is at least 4.

Proof: Among lines containing at least one marked square let the upmost row be $S_1$, the lowmost row be $S_2$, the leftmost column be $T_1$ and the rightmost column be $T_2$. If each of $S_1, S_2, T_1, T_2$ contains only one marked square then all these marked squares are different and each of these squares is a one-marked square. If at least two of $S_1, S_2, T_1, T_2$ contain at least two marked squares each, for each line containing at least two marked squares its end marked squares (not intermediate marked squares) are one-marked and therefore there are at least 4 one-marked squares. Finally without loss of generality suppose that $S_1$ contains at least two marked squares and $S_2$ contains one marked square. Then two end marked squares (not intermediate marked squares) of $S_1$ and the marked square of $S_2$ are one-marked squares. Since $4n-8\geq n+2$ there are at least two marked squares not lying $S_1$. Let the only marked square of $S_2$ be $A$. Let the row containing at least one marked square and closest to $S_2$ be $S'$. If $S'$ contains the only marked square then this marked square is one-marked. Otherwise all end marked squares of $S'$ which are not in the same column with $A$ are one-marked. Thus, there are at least 4 one-marked squares. Done.

By Lemma 2 and Lemma 1 there are at most $4n-8$ marked squares. We are done.