Bilkent University
Department of Mathematics

Problem Of The Month

May 2025

Problem:

Let n be a positive integer and S_n be the set of all positive integers not exceeding n and relatively prime to n. Let $f(n)$ be the smallest positive integer for which the set S_n can be partitioned into $f(n)$ disjoint subsets, such that each of these subsets is an arithmetic progression. Show that there are infinitely many pairs (a, b) such that $a, b > 2025$, $a|b$ and $f(a) \not| f(b)$.

Solution:

Let us show that each pair $(a, b) = (2p, 4p)$, where $p > 1012$ is a prime number satisfies problem conditions.

The set S_{2p} consists of all odd integers from the interval $[1, 2p-1]$ except p. Obviously $f(2p) > 1$. Then since

$$S_{2p} = \{1, 3, \ldots, p-2\} \cup \{p+2, p+4, \ldots, 2p-1\}$$

we get $f(2p) = 2$.

The set S_{4p} consists of all odd integers from the interval $[1, 4p-1]$ except p and $3p$. Obviously $f(4p) > 2$. Then since

$$S_{2p} = \{1, 3, \ldots, p-2\} \cup \{p+2, p+4, \ldots, 3p-2\} \cup \{3p+2, p+4, \ldots, 4p-1\}$$

we get $f(4p) = 3$.

Since $2p \mid 4p$ and $2 \not| 3$ we are done.