Bilkent University
Department of Mathematics

Problem Of The Month

April 2025

Problem:

Is there an infinite sequence $\{a_i\}_{i=1}^{\infty}$ of positive real numbers such that

$$\sum_{i=1}^{n} a_i \geq n^2 \quad \text{and} \quad \sum_{i=1}^{n} a_i^2 \leq n^3 + 2025n$$

for each positive integer n?

Solution: Answer: There is no such a sequence.

Suppose that there is a sequence satisfying conditions. Let us show that there are infinitely many $n \in \mathbb{Z}^+$ such that $a_n > n$. Assume the contrary. Then there exists a constant N such that $a_n \leq n$ for all $n > N$. Hence

$$n^2 \leq \sum_{i=1}^{n} a_i \leq \frac{n^2 + n}{2} + S$$

where first inequality is a problem condition and $S = a_1 + \ldots + a_N$. But this inequality is not correct for sufficiently large values of n, a contradiction.

Now, consider the sequence

$$T_n = n \left(\sum_{i=1}^{n} a_i^2 \right) - \left(\sum_{i=1}^{n} a_i \right)^2.$$

By problem conditions we get $T_n \leq 2025n^2$ for all positive integers n. On the other hand, $T_n = \sum_{1 \leq i < j \leq n} (a_i - a_j)^2$. Let $A = 2\sum_{i=1}^{2026} a_i$ and $B = \sum_{i=1}^{2026} a_i^2$ and n be an integer satisfying $n > A$ and $a_n > n$. Then,

$$2025n^2 \geq T_n \geq (a_n - a_1)^2 + \ldots + (a_n - a_{2026})^2 > 2026n^2 - An + B$$

which is not incorrect for $n > A$, a contradiction.