Bilkent University
Department of Mathematics

Problem Of The Month

March 2025

Problem:

A total of 3300 handshakes were made at a party attended by 600 people. It was observed that the total number of handshakes among any 300 people at the party is at least N. Find the largest possible value for N.

Solution: Answer: 750.

Let us construct an example for $N = 750$. We divide people attending the party to 50 groups each consisting of 12 people. Suppose that handshakes were made only between all persons belonging to the same group. Then the total number of handshakes is $\binom{12}{2} \cdot 50 = 3300$, as required. Suppose that arbitrary 300 persons are chosen by taking $k_1, k_2, \ldots, k_{50}$ persons from 50 groups. Then the total number of handshakes between this 300 people is

$$\binom{k_1}{2} + \cdots + \binom{k_{50}}{2} = \frac{k_1^2 - k_1}{2} + \cdots + \frac{k_{50}^2 - k_{50}}{2} = \frac{k_1^2 + \cdots k_{50}^2 - 750}{2}.$$

By qaudratic-arithmetic mean inequality we get

$$\frac{k_1^2 + \cdots k_{50}^2 - 300}{2} \geq \frac{300^2/50 - 300}{2} = 750,$$

as required.

Now suppose that there is an example for $N > 750$. For any group X of size 300 let $h(X)$ be the total number of handshakes among people of X. Let A be a group such that $h(A) \leq h(X)$ for any other group X and B be the group consisting of the remaining 300 people. For each $a \in A$ and $b \in B$ let $f(a)$ be the total number of handshakes made by a inside A and $g(b)$ be the total number of handshakes made between b and persons from A. By definitions, for all pairs (a, b) we have $f(a) \leq g(b)$. If $g(b) \geq 6$ for each $b \in B$ then the total number of handshakes is at least $N + N + 6 \cdot 300 > 3300$. If $g(b) \leq 5$ for some $b \in B$ then $f(a) \leq 5$ for each $a \in A$ and hence the total number of handshakes inside A is at most $\dfrac{300 \cdot 5}{2} = 750$, a contradiction.