Bilkent University
Department of Mathematics

Problem Of The Month

February 2025

Problem:

At the beginning there are 111 distinguishable balls on the red table and the white table is empty. Starting with a red table we *alternatively* apply moves to the tables. In a move applied to a given table we choose some collection of balls from this table and transfer them to the other table. Find the maximal possible number of moves if each ball collection can be chosen at most once.

Solution: Answer: $2^{111} - 2$.

Let us show that the total number of moves can not be $2^{11} - 1$. If the total number of moves is $2^{111} - 1$ then all non-empty subsets of 111 element set should be selected. The total number of subsets containing given element is 2^{110}, which is an even number. Therefore, at the end of all moves all balls should be on the red table. On the other hand, since $2^{111} - 1$ is odd, the white table should contain at least one ball, a contradiction.

Suppose that at the beginning the red table contains n balls $\{a_1, a_2, \ldots, a_n\}$ and the white table is empty. We show that the total number of moves can be equal to $2^{111} - 2$.

By induction over n, let us show that for each $n \geq 2$, it is possible to make $2^n - 2$ moves such that the only not made move is $\{a_1\}$ and after making all $2^n - 2$ moves, all balls except a_1 are on the red table and a_1 is on the white table.

If $n = 2$ it can be done by applying moves $\{a_1, a_2\}, \{a_2\}$.

Now suppose that for $n = k$ the inductive hypothesis is correct and $n = k + 1$. Let us separate ball a_{k+1} and apply $2^k - 2$ moves to remaining $n = k$ balls by inductive hypothesis. After that make a move $\{a_{k+1}\}$ transferring ball a_{k+1} to the white table. Then by inductive hypothesis the white table contains only two balls: a_1 and a_{k+1}. Let us make a move $\{a_1, a_{k+1}\}$. Finally we can repeat all $2^k - 2$ moves made at the beginning by adding a ball a_{k+1} to all moves. In total we will make $2^k - 2 + 2 + 2^k - 2 = 2^{k+1} - 2$ legal moves. Done.