Bilkent University
Department of Mathematics

Problem Of The Month

December 2024

Problem:

Let S be a set consisting of 31 positive real numbers. For each non-empty subset $A \subset S$ let $f(A)$ be the product of all elements of A. We say that a subset $A \subset S$ is *rational* if $f(A)$ is a rational number. We say that a subset $A \subset S$ is *irrational* if $f(A)$ is an irrational number. Is there any set S having exactly 2023 rational subsets? Is there any set S having exactly 2025 irrational subsets?

Solution:

Let us show that there is a set S having exactly 2023 rational subsets. Let n be a positive integer whose value will be determined later on. Let S be the set consisting of $2^{\frac{2^i}{n}}$ for $i = 0, 1, \ldots, 30$. Let Q be the set of subsets of S whose product of own elements is a rational number together with the empty set. Since every positive integer has a unique representation as sum of distinct powers of 2, the cardinality of Q is precisely the number integers from 0 to $N = 2^{31} - 1$ which are divisible by n. Therefore,

$$|Q| = 2023 \iff 1 + \left\lfloor \frac{N}{n} \right\rfloor = 2023 \iff$$

$$2022 \leq \frac{N}{n} < 2023 \iff \frac{N}{2022} < n \leq \frac{N}{2023}.$$

Thus, if $N/2022 - N/2023 \geq 1$, then there exists an integer n such that $|Q| = 2023$. As $2022 \cdot 2023 < (2^{11})^2 < 2^{23} - 1 = N$, there is such an n.

Let us show that there is no set S having exactly 2025 irrational subsets. If all elements of S are rational then S has $2^{31} - 1 > 2025$ rational subsets. Let $e \in S$ be an irrational element of S. Then for each subset $A \subset S$ either A or $A \subset S$ is irrational. Thus there are $2^{30} > 2025$ irrational subsets of S.