Bilkent University
Department of Mathematics

Problem Of The Month

October 2024

Problem:

We say that a positive integer k is *nice* if there is a positive integer n having exactly k positive divisors $d_1, \ldots, d_k$ such that $d_i \not\equiv d_j \pmod{k+1}$ for all $i \neq j$. Find all nice numbers.

Solution: Answer: $k = 3$ and $k = p - 1$, where p is prime.

We consider two cases.

Case 1: 0 is one of the remainders $d_i \pmod{k+1}$.

Then $k+1$ divides exactly one of the divisors. Thus $k+1$ divides n and indeed $k+1 = n$. So $k+1$ has k positive divisors. Then $k | k+1$ or $k-1 | k+1$. Therefore, $k = 1, 2$ or 3.

Case 2: 0 is not one of the remainders.

Suppose that $k+1$ is not prime. Then $k+1 = pm$ where p is a prime number and $m > 1$ is an integer. The possible remainders $d_i \pmod{k+1}$ which are divisible by p are $p, 2p, \ldots, p(m-1)$. Let $n = p^\alpha r$ where $\gcd(p, r) = 1$. Let s be the number of positive divisors of r. Then $k = (\alpha+1)s$ and $m - 1 = \alpha s$ is the number of positive divisors of n which are divisible by p. Then we have

$$2 < \frac{pm-1}{m-1} = \frac{k}{m-1} = \frac{(\alpha+1)s}{\alpha s} = \frac{\alpha+1}{\alpha} \leq 2,$$

which is a contradiction. Thus, in this case $k+1$ has to be prime.

Now we show that for any given prime number p the number $k = p-1$ is nice. Let g be a primitive root of p. Consider the arithmetic sequence $g + lp$, $l = 1, 2, \ldots$. By Drichlet's theorem this sequence contains a prime number q. Then $n = q^{p-2}$ has exactly $p-1$ divisors with different remainders $\pmod{p}$ and hence $k = p-1$ is nice. Done.