Bilkent University
Department of Mathematics

Problem Of The Month

September 2024

Problem:

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of real numbers defined by

$$a_1 = 30, \quad b_1 = 31 \quad \text{and}$$

$$a_{n+1} - a_n = \frac{1}{b_n}, \quad b_{n+1} - b_n = \frac{1}{a_n} \quad \text{for all} \quad n \geq 1.$$

Find the smallest integer k such that $a_k \geq 60$.

Solution: Answer: 1396.

Since

$$\frac{a_{n+1}}{b_{n+1}} = \frac{a_n + \frac{1}{b_n}}{b_n + \dfrac{1}{a_n}} = \frac{a_n}{b_n}$$

we get that $\dfrac{a_n}{b_n}$ is a constant and hence equals $\dfrac{30}{31}$. Therefore, $a_k > 60$ is equivalent to $b_k > 62$ and $a_k b_k > 60 \cdot 62 = 3720$. Multiplying expressions for a_{n+1} and b_{n+1} we get

$$a_{n+1} b_{n+1} = a_n b_n + 2 + \frac{1}{a_n b_n}.$$

Hence, for any k we have $a_k b_k > 30 \cdot 31 + 2(k-1)$ hence at $k = 1396$ the inequality $a_k b_k > 3720$ holds. Moreover, if $k = 1395$, we have

$$a_k b_k = 1396 - 2 + \sum_{i=0}^{k-1} \frac{1}{a_i b_i} < 1396 - 2 + \frac{1394}{30 \cdot 31} < 1396.$$

Thus, the minimal value of k satisfying conditions is 1396.