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Problem Of The Month

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Problem:

There are 353 red balls, 535 white balls and 888 empty boxes numbered $1, 2, \ldots, 888$ on the table. Alice distributed these 888 balls to 888 boxes so that each box contains one ball. Bob is going to find out ball colours in all 888 boxes. In order to do this Bob writes N pairs (i, j), $1 \leq i < j \leq 888$ to the board. After that for each pair (i, j) on the board Alice informs Bob, whether balls in the boxes i and j are same-coloured. Find the minimal value of N for which Bob can guarantee to determine the ball colours in all 888 boxes.

Solution: Answer: 886.

Let us show that Bob can guarantee to determine the colours of all balls for $N = 886$. Bob writes the pairs $(1, 2), (1, 3), \ldots, (1, 887)$ on the board. Let us assign a number i to the ball contained in the box number i. Suppose that out of balls numbered $2, 3, \ldots, 887$ the number of balls same-coloured with 1 is a and the number of differently coloured balls is $b = 886 - a$. Then the colours of all balls can be easily determined: one of these numbers a and b is greater or equal to 443. If $a \geq 443$ then ball 1 is white. $a + 1$ is either 534 or 535. In the first case ball 888 is white, in the second case ball 888 is red. If $b \geq 443$ then ball 1 is red. b is either 534 or 535. In the first case ball 888 is white, in the second case ball 888 is red.

Now we show that if $N \leq 885$ than Bob can not guarantee to determine colours of all balls. Suppose that N pairs (i, j) are already chosen by Bob. Let us define a graph G consisting of 888 vertices representing balls where vertices i and j are connected by an edge if and only if there is a pair (i, j) on the board. Since the total number of edges is $N \leq 885$ the graph G has at least 3 connected components. Alice divides these connected components into 3 non-empty groups C_1, C_2 and C_3 with number of vertices $|C_1|$, $|C_2|$ and $|C_3|$. Assuming $|C_1| \geq |C_2| \geq |C_3|$ we get that $|C_2| + |C_3| \leq 592$. Obviously at least one of groups C_2, C_3 and $C_2 \cup C_3$ contains even number of vertices. Alice can divide vertices of this group into two parts A_1 and A_2, where $|A_1| = |A_2| < 296$. Since Bob is going to guarantee to determine ball colours we can suppose that Alice knows all pairs on the board before her ball distribution. Thus, Alice can make all balls in A_1 red coloured and all balls in A_2 white coloured or alternatively all balls in A_1 white coloured and all balls in A_2 red coloured and the remaining balls will have arbitrary colours just to provide 353 red and 535 white balls in total. In both cases the answers of Alice will be the same. Therefore, Bob can not determine the ball colours of this group containing even number of boxes. Done.