Bilkent University
Department of Mathematics

Problem Of The Month

May 2024

Problem:

Find the maximal possible value of

$$\sum_{i,j=1,2,\ldots,n,\ i\neq j} \lceil x_i x_j \rceil - (n-1)\left(\sum_{i=1,2,\ldots,n} \lfloor x_i^2 \rfloor\right)$$

for all real numbers $x_1, x_2, \ldots, x_n$.

Note: For a real number x, $\lceil x \rceil$ is the smallest integer which is not less than x and $\lfloor x \rfloor$ is the largest integer not exceeding x: $\lceil 1.8 \rceil = 2$, $\lfloor 2.4 \rfloor = 2$ and $\lceil 3 \rceil = \lfloor 3 \rfloor = 3$.

Solution: Answer: $n^2 - n + \lfloor \frac{n^2}{4} \rfloor$.

Let us give an example for the equality case: For $n = 2k$, where k is an integer when k numbers are 0.9 and k numbers are 2.1. For $n = 2k+1$ where k is an integer when k numbers are 0.9 and $k+1$ numbers are 2.1.

Let us rewrite the main expression as

$$S = \sum_{i<j} \left(2\lceil x_i x_j \rceil - \lfloor x_i^2 \rfloor - \lfloor x_j^2 \rfloor\right).$$

For all pairs of real numbers a, b we have

$$2\lceil ab \rceil - \lfloor a^2 \rfloor - \lfloor b^2 \rfloor < 2ab + 2 - (a^2 - 1) - (b^2 - 1) = 4 - (a-b)^2 \leq 4.$$

Since the left hand side is an integer, we get $2\lceil ab \rceil - \lfloor a^2 \rfloor - \lfloor b^2 \rfloor \leq 3$, where the inequality turns to equality say for $a = 1.2$ and $b = 0.9$. Also if $\lfloor a^2 \rfloor - \lfloor b^2 \rfloor$ is even then since the left hand side is also even we get $2\lceil ab \rceil - \lfloor a^2 \rfloor - \lfloor b^2 \rfloor \leq 2$, where the inequality turns to equality when $a = b$ and both squares are not integers.

For real numbers $x_1, x_2, \ldots, x_n$, suppose that $\lfloor x_i^2 \rfloor$ is even for k numbers and $\lfloor x_i^2 \rfloor$ is odd for the remaining $l = n - k$ numbers. Then the contribution of each pair out of $\binom{k}{2}$ pairs of even numbers and $\binom{l}{2}$ pairs of odd numbers to S will be at most 2. The contribution of each pair out of kl pairs of one even and one odd number to S will be at most 3. Therefore,

$$S \leq 3kl + k^2 - k + l^2 - l = (k+l)^2 + kl - k - l = n^2 - n + kl \leq n^2 - n + \lfloor \frac{n^2}{4} \rfloor$$

and we are done.