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PROBLEM OF THE MONTH

February 2024

**Problem:**

The country has 6543 cities and between any two of these cities exactly one of 15 air companies is running two way flights. Find the minimal value of  $n$  such that for any arrangement of all flights one can choose  $n$  air companies such that only by using of flights of these chosen  $n$  companies starting from any city one can reach any other city.

**Solution:** Answer: 8.

Let us divide 15 air companies into groups A and B, consisting of 8 and 7 companies, respectively. Assume that by using of companies of group A some city X is not reachable from some other city Y. It means that the direct flight between X and Y is operated by some air company from B, and also any other city Z is directly connected to either X or Y by some air company from B. Therefore, by using of flights of group B any city is reachable from any other city. Thus, one can choose 8 air companies guaranteeing total connection in the country.

Now we give an example for showing that 7 air companies may not be enough for guaranteeing of total connection. There are  $\binom{15}{7} = 6435$  possible choices of 7 air companies. To each of these choices we assign a city so that for different cities assigned cities are also different. Since  $6435 < 6543$  such one-to-one correspondence is possible. We organize flights such that for each collection of 7 companies all flights from the city assigned to these 7 companies will be operated by the remaining 8 air companies. This is possible because since  $8 + 8 > 15$ , for any two cities their collections of allowed 8 air companies have non-empty intersection and consequently the air company operating between these two cities can be properly chosen. By construction for any choice of 7 air companies the city assigned to this choice is not reachable from other cities. We are done.