Bilkent University
Department of Mathematics

Problem Of The Month

September 2023

Problem:

Let k be a positive integer and $\mathcal{S}$ be a family of 63 sets, each having size k. Suppose that for all $A, B \in \mathcal{S}$, $A \neq B$ we have $A \triangle B \in \mathcal{S}$. Find all possible values of k.

Note: $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Solution: Answer: $k = 32m$, where m is a positive integer.

Let us fix some $A \in \mathcal{S}$ and $x \in A$. Let $\mathcal{S}_x$ be the set of all sets from $\mathcal{S}$ containing x and $\bar{\mathcal{S}}_x = \mathcal{S} \setminus \mathcal{S}_x$ be its complement. It can be readily seen that for any two distinct $B \in \mathcal{S}_x$ and $C \in \mathcal{S}_x$ we have $A \triangle B \in \bar{\mathcal{S}}_x$ and $A \triangle C \in \bar{\mathcal{S}}_x$ and these two sets are distinct. On the other hand, since $A \triangle D \in \mathcal{S}_x$ for any $D \in \bar{\mathcal{S}}_x$ and $A \triangle (A \triangle D) = D$ we get that $|\mathcal{S}_x \setminus A| = |\bar{\mathcal{S}}_x|$. Therefore, since $|\mathcal{S}| = 63$ we get $|\mathcal{S}_x| = 32$. Let $N = |\bigcup_{U \in \mathcal{S}} U|$. Then by counting the total number of all possible pairs (y, U), where $y \in U$ in two different ways we get that $32N = 63k$. Therefore, $32 | k$ and $k = 32m$.

Now we give an example for $k = 32m$. Let $k = 32m$ for some $m \in \mathbb{N}$, and let $L = \{1, 2, \ldots, 6\}$. Let us define 6 sets $T_1, T_2, \ldots, T_6$ such that for any non-empty $R \subseteq L$ the set

$$T_R = \left(\bigcap_{r \in R} T_r \right) \setminus \left(\bigcup_{r \in L \setminus R} T_r \right)$$

contains exactly m elements.

It can be readily seen that the operation of symmetric difference between several sets is commutative and associative. Therefore, the expression $\triangle_{j \in J} T_j$ is well defined. Now, it can be readily seen that for every non-empty $J \subseteq \{1, 2, \ldots, 6\}$, we have

$$\triangle_{j\in J} T_j = \bigcup_{R\subseteq L, |R\cap J| \text{ is odd}} T_R.$$

Since for any non-empty J there are exactly 2^5 subsets $R \subseteq L$ for which $|R \cap J|$ is odd, it follows that $|\triangle_{j\in J} T_j| = 32m$.

By definitions, for any two distinct non-empty J_1 and J_2 the corresponding sets $\triangle_{j\in J_1} T_j$ and $\triangle_{j\in J_2} T_j$ are also distinct. On the other hand,

$$(\triangle_{j\in J_1} T_j) \triangle (\triangle_{j\in J_2} T_j) = \triangle_{j\in J_1 \triangle J_2} T_j.$$

Therefore, the set

$$\mathcal{S} = \{\triangle_{j\in J} T_j : J \subseteq L, J \neq \emptyset\}$$

containing $2^6 - 1 = 63$ elements satisfies all required conditions.