Bilkent University
Department of Mathematics

Problem Of The Month

July-August 2023

Problem:

Find the minimal value of

$$\frac{(a^2+b^2+2c^2+3d^2)(2a^2+3b^2+6c^2+6d^2)}{(a+b)^2(c+d)^2}$$

where a, b, c, d are positive real numbers.

Solution: Answer: 6 .

It follows from Cauchy-Schwarz inequality that

$$(a^2+b^2+2c^2+3d^2)(6c^2+6d^2+3b^2+2a^2)$$

$$\geq 6(ad+bc+ac+bd)^2 = 6(a(c+d)+b(c+d))^2 = 6(a+b)^2(c+d)^2$$

Therefore,

$$\frac{(a^2+b^2+2c^2+3d^2)(2a^2+3b^2+6c^2+6d^2)}{(a+b)^2(c+d)^2} \geq 6.$$

Now note that the value 6 achieves when

$$(a,b,c,d) = \left(\sqrt[4]{6}, \sqrt[4]{\frac{27}{2}}, \frac{\sqrt{6}}{2}, 1\right).$$

We are done.