Bilkent University
Department of Mathematics

Problem Of The Month

April 2023

Problem:

Find the smallest value of

$$xy^3z^2 + \frac{4z}{x} - 8yz - \frac{4}{xy}$$

where x, y, z are positive real numbers satisfying at least one of the following inequalities:

$$xy > \frac{1}{2} \quad \text{and} \quad yz > 1.$$

Solution: Answer: -8. This value is attained at $xy = 1$ and $yz = 2$.

By AM-GM inequality we get

$$2y - \frac{1}{x} \leq xy^2 \ \text{ and } \ 4z - \frac{4}{y} \leq yz^2 \qquad (1)$$

Since at least one of the inequalities: $xy > \frac{1}{2}$ and $yz > 1$ is held at least one of the inequalities

$$2y - \frac{1}{x} > 0 \ \text{ and } \ 4z - \frac{4}{y} > 0$$

is also held. Therefore, we can multiply two inequalities in (1) and

$$-8 - \frac{4z}{x} + 8yz + \frac{4}{xy} = (2y - \frac{1}{x})(4z - \frac{4}{y}) \leq xy^3z^2.$$

Thus,

$$xy^3z^2 + \frac{4z}{x} - 8yz - \frac{4}{xy} \geq -8.$$