Bilkent University
Department of Mathematics

Problem Of The Month

March 2023

Problem:

Any two pupils in the school are either friends or not, the friendship is mutual. For each integer $1 \leq \ell \leq 111$ there is a school pupil having exactly ℓ friends in the school. Given that there is no triple of school pupils such that any two of them are friends, find the minimal possible number of pupils in this school.

Solution: Answer: 167.

Let us replace 111 by k and show that if N is the total number of pupils in the school then $N \geq \frac{3k}{2}$. Suppose that A has friends $B_1, B_2, \ldots, B_k$. Since there is no triple of pupils each two being friends no two pupils among $B_1, B_2, \ldots, B_k$ are friends. Therefore, if for some $1 \leq m \leq k$, B_m has at least $\frac{k}{2}$ friends then the total number of pupils is at least

$$k + 1 + \frac{k}{2} - 1 = \frac{3k}{2}.$$

Otherwise for each $1 \leq i \leq k$, B_i has less than $\frac{k}{2}$ friends. Then the pupils having at least $\frac{k}{2}$ and at most $k - 1$ friends are not among friends of A and the total number of pupils is at least

$$k + 1 + k - \frac{k}{2} - 1 = \frac{3k}{2}$$

Let us give an example for $N = 167$ for $k = 111$:

Let the pupils be $A_1, \ldots, A_{56}$ and $B_1, \ldots, B_{111}$. If for all pairs (i, j) satisfying $1 \leq i \leq 56$, $1 \leq j \leq 111$, $i \leq j$ the pupils A_i and B_j are friends then all conditions are fulfilled.