Bilkent University
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Problem Of The Month

January 2023

Problem:

Find the maximal possible number of ordered pairs (i, j) satisfying

$$\frac{a_i^2}{4} + a_j \geq \frac{1}{2022}$$

where $a_1, a_2, \ldots, a_{2023}$ are non-negative real numbers satisfying $a_1 + a_2 + \ldots + a_{2023} = 1$.

Solution: Answer: $2023^2 - 2023$.

Let us replace 2023 with n and solve the problem for all $n \geq 2$. When $a_1 = a_2 = \cdots = a_{n-1} = \dfrac{1}{n-1}$ and $a_n = 0$ the total number of pairs satisfying the inequality is $n^2 - n$.

We will show that for all $a_1, a_2, \ldots, a_n$ are non-negative real numbers satisfying $a_1 + a_2 + \ldots + a_n = 1$ there are at least n pairs with $\dfrac{a_i^2}{4} + a_j < \dfrac{1}{n-1}$. Without loss of generality assume that $a_1 \geq a_2 \geq \ldots \geq a_n$. Let us consider the pairs

$$(i, j) = (1, n), (2, n-1), \ldots, (n, 1).$$

If the the inequality is not held for one of these pairs, say (i, j), then by increasing the indices i and j we get n required pairs also not satisfying the inequality. Now suppose that for all pairs these pairs the inequalities are held:

$$\frac{a_1^2}{4} + a_n \geq \frac{1}{n-1}, \ \frac{a_2^2}{4} + a_{n-1} \geq \frac{1}{n-1}, \ \ldots, \ \frac{a_n^2}{4} + a_1 \geq \frac{1}{n-1}.$$

Since the sum of the $a_1 + a_n, a_2 + a_{n-1}, \ldots, a_n + a_1$ is equal to 2, $a_k + a_{n+1-k} \leq \dfrac{2}{n}$ for some k. Let $a_k = p$ and $a_{n+1-k} = q$. Then $p + q \leq \dfrac{2}{n}$ and

$$\frac{p^2}{4}+q \geq \frac{1}{n-1}, \quad \frac{q^2}{4}+p \geq \frac{1}{n-1}.$$

By multiplying these two inequalities we get

$$(\frac{p^2}{4}+q)(\frac{q^2}{4}+p) = \frac{p^3+q^3}{4}+\frac{p^2q^2}{16}+pq \geq \frac{1}{(n-1)^2}. \qquad (1)$$

Since $p+q \leq \frac{2}{n}$ we have $pq \leq \frac{p+q)^2}{4} \leq \frac{1}{n^2}$ ve $p^3+q^3 \leq (p+q)^3 \leq \frac{8}{n^3}$. Then by using these inequalities in (1) we get

$$\frac{2}{n^3}+\frac{1}{16n^4}+\frac{1}{n^2} \geq \frac{1}{(n-1)^2}.$$

But since

$$\frac{2}{n^3}+\frac{1}{16n^4}+\frac{1}{n^2} < \frac{2}{n^3}+\frac{1}{n^4}+\frac{1}{n^2} = \frac{(n+1)^2}{n^4} < \frac{1}{(n-1)^2}.$$

we get a contradiction with (1). Thus, there are n pairs (i,j) not satisfying $\frac{a_i^2}{4}+a_j \geq \frac{1}{n-1}$. Done.