Bilkent University
Department of Mathematics

Problem Of The Month

November 2022

Problem:

For each pair (p, a), where p is a prime number and a is a positive integer the sequences $\{a_n\}$ and $\{b_n\}$ are defined as

$$a_1 = a \text{ and } a_{n+1} = a_n + p\lfloor \sqrt[p]{a_n} \rfloor$$

$$b_n = \sqrt[p]{a_n}$$

A pair (p, a) is said to be *good* if the sequence $\{b_n\}$ contains infinitely many integers. Find all values of p such that all pairs (p, a), $a = 1, 2, \dots$ are good.

Solution: Answer: For all primes p all pairs (p, a), $a = 1, 2, \dots$ are good.

Let p be a fixed prime number and a be a given positive integer. Let m be a positive integer with $m^p > a$. Since the sequence $\{a_n\}$ is increasing the exists a smallest term a_k of the sequence $\{a_n\}$ satisfying $a_k > m^p$. By definitions, $a_{k-1} < m^p$ and

$$a_k - a_{k-1} = p\lfloor \sqrt[p]{a_{k-1}} \rfloor < pm.$$

Therefore, $a_k = m^p + r$, where $r < pm$. Let us show that along with $a_k = m^p + r$ the number $(m+1)^p + r - 1$ is also an element of the sequence $\{a_n\}$. Indeed, since p is a prime number the difference

$$((m+1)^p + r - 1) - (m^p + r) = \binom{p}{p-1} m^{p-1} + \binom{p}{p-2} m^{p-2} + \cdots + \binom{p}{1} m$$

is a multiple of pm. Therefore, the smallest term a_l of the sequence $\{a_n\}$ satisfying $a_l > (m+1)^p$ has a form $a_l = (m+1)^p + r - 1$. By applying this argument repeatedly one can show that the sequence $\{a_n\}$ contains terms $(m+2)^p + r - 2$, $(m+3)^p + r - 3, \dots, (m+r-1)^p + 1$ and finally the term $a_t = (m+r)^p$. Thus, $b_t = m + r$ is an integer. By choosing sparsely located distinct numbers m we can get infinitely many integer terms of $\{b_n\}$.