Bilkent University
Department of Mathematics

Problem Of The Month

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Problem:

Let $\mathbb{Q}$ and $\mathbb{Q}^+$ be the set of all rational and all positive rational numbers, respectively. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}$ satisfying

$$f(x) + f(y) = \left(f(x+y) + \frac{1}{x+y}\right)(1 - xy + f(xy)) \qquad (*)$$

for all $x, y \in \mathbb{Q}^+$.

Solution: Answer: $f(x) = x - \dfrac{1}{x}, \ \forall x \in \mathbb{Q}^+$.

We will prove three lemmas.

Lemma 1. $f(1) = 0$.

Proof: By putting $x = y = 1$ to $(*)$ we get $2f(1) = \left(f(2) + \dfrac{1}{2}\right) f(1)$. Assume that $f(1) \neq 0$. Then readily $f(2) = \dfrac{3}{2}$. By putting $x = y = 2$ to $(*)$ we get $3 = 2f(2) = \left(f(4) + \dfrac{1}{4}\right)(f(4) - 3)$. Therefore, either $f(4) = \dfrac{15}{4}$ or $f(4) = -1$. By putting $y = 1$ to $(*)$ we get $f(x) + f(1) = \left(f(x+1) + \dfrac{1}{x+1}\right)(1 - x + f(x))$. Taking $x = 2, 3, 4, 5$ in the last equation we get

$$\frac{3}{2} + f(1) = \left(f(3) + \frac{1}{3}\right) \cdot \frac{1}{2} \qquad (1)$$

$$f(3) + f(1) = \left(f(4) + \frac{1}{4}\right)(f(3) - 2 \qquad (2)$$

$$f(4)+f(1)=\left(f(5)+\frac{1}{5}\right)(f(4)-3) \qquad (3)$$

$$f(5)+f(1)=\left(f(6)+\frac{1}{6}\right)(f(5)-4) \qquad (4)$$

If $f(4)=-1$, (1) and (2) yield $f(1)=-\frac{19}{27}$ and $f(3)=\frac{34}{27}$, and (3) and (4) yield $f(5)=\frac{61}{270}$, $f(6)=-\frac{245}{6114}$. Finally by putting $x=2, y=3$ to $(*)$ we get

$$f(2)+f(3)=\left(f(5)+\frac{1}{5}\right)(f(6)-5).$$

This relationship is not held for values of $f(2), f(3), f(5), f(6)$ found above. Hence the only possibility is $f(4)=\frac{15}{4}$. In this case we have

$$f(3)+f(1)=4(f(3)-2), \quad \frac{3}{2}+f(1)=\left(f(3)+\frac{1}{3}\right)\cdot\frac{1}{2}.$$

Solving last two equations we get $f(1)=0$. Done.

Lemma 2. $f(2)=\frac{3}{2}$.

Proof: Since $f(1)=0$, for all $x\in\mathbb{Q}^+$ we have

$$f(x)=\left(f(x+1)+\frac{1}{x+1}\right)(1-x+f(x)) \qquad (5)$$

Taking $x=2,3$ in (5), we get

$$f(2)=\left(f(3)+\frac{1}{3}\right)(f(2)-1), \quad f(3)=\left(f(4)+\frac{1}{4}\right)(f(3)-2).$$

By putting $x=2, y=2$ to $(*)$ we get $2f(2)=\left(f(4)+\frac{1}{4}\right)(f(4)-3)$. Last three equations yield a cubic equation in terms of $t=f(4)$ as given below:

$$16t^3-32t^2-101t-15=0.$$

$t=f(4)$ should be rational as the range of f is rational numbers. The only rational root of this equation is $t=\frac{15}{4}$. Then it readily follows that $f(3)=\frac{8}{3}$ and $f(2)=\frac{3}{2}$.

Lemma 3. $f(n)=n-\frac{1}{n}$ for all positive integers n.

Proof: $f(1) = 0$ for $n = 1$. Proof for $n \geq 2$ readily follows from (5) by induction over n. Done.

Finally we prove that for all $x = \dfrac{m}{n}$

$$f(m/n) = \frac{m}{n} - \frac{n}{m}\,. \tag{6}$$

Proof will be carried out by induction over $m \geq 1$. In the base case $m = 1$ by putting $y = \dfrac{1}{x}$ to $(*)$ we get that for all $x \in \mathbb{Q}^+$

$$f(x) + f\left(\frac{1}{x}\right) = 0$$

and by Lemma 3 for all positive integers n we obtain the required formula

$$f\left(\frac{1}{n}\right) = \frac{1}{n} - n.$$

Now suppose that (6) is correct for m. By putting $x = \dfrac{m}{n}$, $y = \dfrac{1}{n}$ to $(*)$ we get

$$f(m/n) + f(1/n) = \left(f((m+1)/n) + \frac{n}{m+1}\right)\left(1 - \frac{m}{n^2} + f(m/n^2)\right). \tag{7}$$

Putting $f(m/n) = \dfrac{m}{n} - \dfrac{n}{m}$ and $f(m/n^2) = \dfrac{m}{n^2} - \dfrac{n^2}{m}$ to (7) and

simplifying we get the required formula for $m + 1$:

$$f\left(\frac{m+1}{n}\right) = \frac{m+1}{n} - \frac{n}{m+1}.$$

We are done.