**Bilkent University**
**Department of Mathematics**

# Problem Of The Month

April 2022

**Problem:**

For a polynomial $Q$ with integer coefficient and prime $p$, we say that $Q$ excludes $p$ if there is no integer $n$ for which $p \mid Q(n)$. Does there exist a polynomial with integer coefficients having no rational roots which excludes exactly one prime?

**Solution:** Answer: Yes, for example $Q(x) = (2x^3+1)(x^2-x+1)$ satisfies all conditions.

Clearly $Q(x)$ has no rational roots. We will show that the only prime excluded by $Q(x)$ is $p = 2$.

*Observation 1*: For any prime $p$ satisfying $p \equiv 1 \pmod 3$ there exists an integer $n$ such that $n^2 - n + 1 \equiv 0 \pmod p$.

*Proof.* Let $\omega$ be a primitive root modulo $p$ and $n = -\omega^{(p-1)/3}$. Then $n^3 \equiv -1 \pmod p$ and $n \not\equiv -1 \pmod p$. Now since $n^3+1 = (n+1)(n^2-n+1)$ we get $n^2-n+1 \equiv 0 \pmod p$.

*Observation 2*: For any odd prime $p$ satisfying $p \equiv 2 \pmod 3$ there exists an integer $n$ such that $2n^3 + 1 \equiv 0 \pmod p$.

*Proof.* Let $a$ be an integer satisfying $2a + 1 \equiv 0 \pmod p$. Since $3 \mid p - 2$ we get $3 \mid 2p - 1$. Then for the integer $n = a^{(2p-1)/3}$ by Fermat's little theorem we have $n^3 \equiv a^{2p-1} \equiv a \pmod p$. Therefore, $2n^3 + 1 \equiv 2a + 1 \equiv 0 \pmod p$.

$Q(x)$ does not exclude $p = 3$ since $3 \mid Q(2)$. By Observation 1 any prime $p \equiv 1 \pmod 3$ is not excluded by $Q(x)$. By Observation 2 any odd prime $p \equiv 2 \pmod 3$ is not excluded by $Q(x)$. Since both $2n^3+1$ and $n^2-n+1$ are always odd numbers $p = 2$ is excluded by $Q(x)$. We are done.