Bilkent University
Department of Mathematics

Problem Of The Month

March 2022

Problem:

Find all pairs (p, q) od prime numbers satisfying

$$2^p = 2^{q-2} + q!$$

Solution: Answer: $(p, q) = (3, 3), (7, 5)$.

If $q = 2$ then no pair $(p, 2)$ satisfies the equation. If $q = 3$ and $q = 5$ then the only pairs satisfying the equation are $(3, 3)$ and $(5, 7)$, respectively.

Let us show that there is no solution for $q \geq 7$. Consider the binary representation of q: $q = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_r}$ where $0 \leq a_1 < a_2 < \cdots < a_r$ are integers and r is the number of 1's in the binary representation of q. For all $1 \leq k \leq r$ and $1 \leq i \leq a_k$ the number $\frac{2^{a_k}}{2^i}$ is an integer. Furthermore, when $i > a_k$, we get $\left\lfloor \frac{2^{a_k}}{2^i} \right\rfloor = 0$. Finally, we have $\sum_{i=1}^{\infty} \left\lfloor \frac{2^{a_k}}{2^i} \right\rfloor = 2^{a_k} - 1$. Therefore, $v_2(q!)$, the highest power of 2 in $q!$ can be written as

$$v_2(q!) = \sum_{i=1}^{\infty} \left\lfloor \frac{q}{2^i} \right\rfloor = q - r.$$

The original equation is equivalent to $2^{q-2}(2^{p-q+2} - 1) = q!$, where $p - q + 2 > 0$. Hence $v_2(q!) = q - 2$. Therefore, $r = 2$ and $q = 2^{a_1} + 2^{a_2}$. Since q is a prime number, we get $a_1 = 0$ and $a_2 = 2^t$ for some non-negative integer t (q is a Fermat prime).

As $q \geq 7$ we have $2^{p-q+2} \equiv 1 \pmod 7$ and $p - q + 2 \equiv 0 \pmod 3$. Using the fact $q = 2^{2^t} + 1 \equiv 2 \pmod 3$ we get $3 \mid p$ and hence $p = 3$. For $q \geq 7$ no pair $(3, q)$ satisfies the equation.