Bilkent University
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Problem Of The Month

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Problem:

A positive integer number s is said to be *n-smooth* if $s = a_1^2 + a_2^2 + \cdots + a_n^2$, where each $a_i, i = 1, 2, \ldots, n$ is divisible by n. An integer number s is said to be *n-rough* if $s = a_1^2 + a_2^2 + \cdots + a_n^2$, where each $a_i, i = 1, 2, \ldots, n$ *is not* divisible by n. Find all positive integers n for which any n-smooth number is n-rough number.

Solution: Answer: All positive integers except 1,2 and 4.

A positive integer n is said to be *good* if any n-smooth number is n-rough number. We first show that if n is good, so is any multiple of n. Let $m = nk$ and $x_1, x_2, \ldots, x_m$ be integers such that $m|x_i$ for all $1 \leq i \leq m$. Then since $n|x_i$ for all $1 \leq i \leq m$ and n is good, there exist integers $y_1, y_2, \ldots, y_m$ such that

$$\sum_{i=nl+1}^{n(l+1)} x_i^2 = \sum_{i=nl+1}^{n(l+1)} y_i^2$$

for all $0 \leq l \leq k-1$ and $n \nmid y_i$ for all $1 \leq i \leq m$. Therefore we obtain that

$$\sum_{i=1}^{m=nk} x_i^2 = \sum_{i=1}^{m=nk} y_i^2$$

and $m = nk \nmid y_i$ for all $1 \leq i \leq m$.

Next we show that all positive odd integers are good.

Lemma: Let n be a positive odd integer and $x_1, x_2, \ldots, x_n$ be integers with at least one of them is not divisibly by n. Then there exist integers $y_1, y_2, \ldots, y_n$ such that none of them is divisible by n and

$$\sum_{i=1}^{n} (nx_i)^2 = \sum_{i=1}^{n} y_i^2.$$

Proof: Without loss of generality we may assume that $n \nmid x_1$. Let $X = 2\sum_{i=1}^{n} x_i$. If $n|X$, then replace x_1 by $-x_1$. As $n \nmid x_1$ and n is odd, $n \nmid 4x_1$ and hence we may assume that $n \nmid X$. Then by the following identity

$$\sum_{i=1}^{n}(nx_i)^2 = \sum_{i=1}^{n}(X - nx_i)^2$$

letting $y_i = X - nx_i$ for all $1 \leq i \leq n$ works.

For a positive odd integer n, if a positive integer a is sum of squares of n integers with each of them is divisible by n, then there exist integers $x_1, x_2, \ldots, x_n$ and a positive integer r such that $a = \sum_{i=1}^{n}(n^r x_i)^2$ and $n \nmid x_i$ for some $1 \leq i \leq n$. Applying the lemma r times we can find integers $y_1, y_2, \ldots, y_n$ such that $a = \sum_{i=1}^{n} y_i^2$ and $n \nmid y_i$ for all $1 \leq i \leq n$.

Next we show that 8 is good. Let a be positive integer which is sum of squares of 8 integers with each of them is divisible by 8. Then $64|a$, hence $a \geq 64$ and $a = 1^2 + 4^2 + 4^2 + 4^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2$ for some integers x_1, x_2, x_3, x_4 by Lagrange's four-square theorem. Note that $x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv 7 \ (mod\ 8)$ and the only way to get 7 as sum of four quadratic residues in $(mod\ 8)$ is 1+1+1+4. Therefore, $8 \nmid x_i$ for all $1 \leq i \leq 4$.

Finally, 4 is not good since n-smooth number $32 = 4^2 + 4^2 + 0^2 + 0^2$ is not n-rough. Therefore, 1 and 2 are also not good numbers.