Bilkent University
Department of Mathematics

Problem Of The Month

July-August 2021

Problem:

Find all real numbers c for which there exists a non-constant function $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(x-f(y))=f(x-y)+c(f(x)-f(y))$$

for all real numbers x and y.

Solution: Answer: $c=0$.

For $c=0$ the function $f(x)=x$ satisfies the conditions.

Suppose that $c\neq 0$. By taking $x=y$ in the main equation we get

$$f(y-f(y))=f(0),\ \forall y\in\mathbb{R} \qquad (1)$$

By inserting $y-f(y)$ instead of y in the main equation and by (1) we get

$$f(x-f(0))=f(x-y+f(y))+c(f(x)-f(0)),\ \forall x,y\in\mathbb{R}.$$

By taking $y=0$ in the main equation we get

$$f(x-f(0))=f(x)+c(f(x)-f(0)),\ \forall x\in\mathbb{R}.$$

Last two identities yield

$$f(x-y+f(y))=f(x),\ \forall x,y\in\mathbb{R}.$$

By taking $y=x$ in the last identity we get

$$f(f(x))=f(x),\ \forall x\in\mathbb{R} \qquad (2)$$

By inserting $f(y)$ instead of y in the main equation and by (2) we get

$$f(x - f(y)) = f(x - f(y)) + c(f(x) - f(y)), \ \forall x, y \in \mathbb{R}$$

and consequently

$$c(f(x) - f(y)) = 0, \ \forall x, y \in \mathbb{R}.$$

Therefore, for $c \neq 0$ there is no non-constant function satisfying the conditions.