Bilkent University
Department of Mathematics

Problem Of The Month

June 2021

Problem:

Find all positive integers n for which

$$\frac{20 \cdot 5^n - 2}{3^n + 47}$$

is an integer number.

Solution: Answer: There is no n.

If n is even then $3^n + 47 = (4-1)^n + 47$ is divisible by 4, but $20 \cdot 5^n - 2$ is not, a contradiction.

Suppose that $n = 2k+1$. Since $20 \cdot 5^n - 2$ is not divisible by 5, $3^n + 47$ is also not divisible by 5. Therefore, $3^{2k+1} + 47 \equiv (-1)^k \cdot 3 + 2 \not\equiv 0 \pmod 5$ and k is also odd: $n = 4l + 3$. Then $20 \cdot 5^n - 2 = 100 \cdot 5^{4l+2} - 2$. Now since $100 \cdot 5^{4l+2}$ is a perfect square, 2 should be a quadratic residue modulo p for any odd prime divisor of $20 \cdot 5^n - 2$. It is well known that in this case $p \equiv \pm 1 \pmod 8$. Consequently, any odd prime divisor q of $3^n + 47$ also have a form $q = \pm 1 \pmod 8$. Now since $3^n + 47 = 3^{4l+3} + 47 = 81^l \cdot 27 + 47 \equiv 10 \pmod{16}$ we get $\dfrac{3^n + 47}{2} = 8m + 5$. This contradicts with the fact that $q = \pm 1 \pmod 8$. We are done.