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Problem Of The Month

January 2021

Problem:

There are 2021 distinguishable boxes on the table. Starting Alice, Alice and Bob take turn writing an unordered box pair to the table (each unordered box pair can be written at most once). They stop when there are 4038 written pairs on the table. After that Bob numerates all box pairs by numbers $1, 2, \ldots, 4038$ and for each $k = 1, 2, \ldots, 4038$ puts k balls into each box belonging to the pair numbered k. Can Bob guarantee that any two boxes will contain different number of balls?

Solution: Answer: Yes, Bob can guarantee that any two boxes will contain different number of balls. Suppose that Alice at the first move writes a pair (A_1, A_2). At each move Bob chooses pairs containing box A_1 (if possible). By acting this way he can guarantee that all pairs (A_1, A_i), $i = 2, 3, \ldots, 2021$ will be on the table. Bob numerates all 2018 pairs not containing A_1 randomly by numbers $1, 2, \ldots, 2018$ and accordingly puts balls into these boxes. Let $t(A_i)$ be the total number of balls in the box A_i after this procedure. Without loss of generality, assume that

$$t(A_2) \leq t(A_3) \leq \cdots \leq t(A_{2021})$$

After that Bob for each $i = 2, 3, \ldots, 2021$ numerates (A_1, A_i) by $2017 + i$ and accordingly distributes balls. Hereby we get

$$t(A_2) < t(A_3) < \cdots < t(A_{2021}).$$

For each $i = 2, 3, \ldots 2021$ the box A_i received balls at most 2019 times and only in one case the number of balls was more than 2018. Therefore,

$$t(A_1) = 2019 + 2020 + \cdots + 4038 > 2018 \cdot 2018 + 4038 > t(A_i)$$

and as a result any two boxes contain different number of balls.