Bilkent University
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Problem Of The Month

December 2020

Problem:

Let N be the total number of bijective functions

$$f : \{1, 2, \dots, 2020\} \to \{1, 2, \dots, 2020\}$$

satisfying $f(f(f(k))) = k$ for all $k = 1, 2, \dots, 2020$. Show that N is divisible by 3^{336}.

Solution: Let $M(n)$ be the total number of bijective functions

$$f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$$

satisfying $f(f(f(k))) = k$ for all $k = 1, 2, \dots, n$. It turns out that for $n > 2$

$$M(n + 1) = M(n) + n(n - 1)M(n - 2).$$

Indeed, let $f : \{1, 2, \dots, n+1\} \to \{1, 2, \dots, n+1\}$ be a function satisfying the conditions. Then for $f(n + 1)$ there are two options: either $f(n + 1) = (n + 1)$ or for some t and s $f(n + 1) = t, f(t) = s$ and $f(s) = n + 1$.

Readily $M(1) = 1, M(2) = 1$ and $M(3) = 3$ and then by the recurrent formula $M(4) = 9, M(5) = 21, M(6) = 81$. Let us show that if $M(6p - 2), M(6p - 1), M(6p)$ are divisible by 3^a then all terms $M(i)$ starting $i = 6p + 3$ are divisible by 3^{a+1}. Indeed, if $M(6p) = 3^a K$ then $M(6p + 1) = 3^a K + 6p(6p - 1)M(6p - 2)$ and $M(6p + 2) = M(6p + 1) + (6p + 1)(6p)M(6p - 1)$. Therefore, in $\pmod{3^{a+1}}$ we have $M(6p + 3) \equiv M(6p + 2) + (6p + 2)(6p + 1)M(6p) \equiv 2 \cdot 3^a + 3^a \equiv 0$. Similarly, $M(6p + 4), M(6p + 5)$ and hence all subsequent terms are divisible by 3^{a+1}. Now since $M(4), M(5)$ and $M(6)$ are divisible by 3 we get that $N = M(2020) = M(6 \cdot 336 + 4)$ is divisible by 3^{336}.

Note: The highest power of 3 dividing $M(2020)$ is 450.