Bilkent University
Department of Mathematics

Problem Of The Month

September 2020

Problem:

Suppose that for some real number M and all real numbers x, y, z satisfying $0 < x, y, z < 1$

$$\frac{xyz(x+y+z)+(xy+yz+zx)(1-xyz)}{xyz\sqrt{1-xyz}} \geq M.$$

Find the maximal possible value of M.

Solution: Answer: 6.

By simple calculations, we obtain that

$$S = \frac{xyz(x+y+z)+(xy+yz+zx)(1-xyz)}{xyz\sqrt{1-xyz}} = \sum_{\text{cyc}} \frac{x - yz + \dfrac{1}{x}}{\sqrt{1-xyz}}.$$

By AM-GM inequality, we get

$$\sqrt{1-xyz} = \sqrt{x\left(\frac{1}{x} - yz\right)} \leq \frac{1}{2}\left(x - yz + \frac{1}{x}\right),$$

or equivalently,

$$\frac{x - yz + \dfrac{1}{x}}{\sqrt{1-xyz}} \geq 2.$$

The sum of three cyclic versions of the last inequality yields $S \geq 6$.

The equality holds when $x = \frac{1}{x} - yz$, $y = \frac{1}{y} - xz$ and $z = \frac{1}{z} - xy$ which gives $x = y = z = t$ where t is the unique real root of the equation $t^3 + t^2 = 1$.