Bilkent University
Department of Mathematics

Problem Of The Month

July-August 2020

Problem:

Let $A_1A_2A_3A_4$ be a circumscribed quadrilateral with the perimeter p_1 and with the sum of its diagonals k_1 and let $B_1B_2B_3B_4$ be a circumscribed quadrilateral with the perimeter p_2 and with the sum of its diagonals k_2. Given

$$p_1^2 + p_2^2 = (k_1 + k_2)^2$$

prove that $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are congruent squares.

Solution:

Lemma. Let $ABCD$ be a convex quadrilateral. Then

$$(AB + CD)^2 + (BC + AD)^2 \geq (AC + BD)^2,$$

and equality holds when $ABCD$ is a rectangle.

Proof: By Ptolemy's inequality

$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD$$

and by parallelogram inequality

$$AB^2 + BC^2 + CD^2 + AD^2 \geq AC^2 + BD^2.$$

Therefore,

$$(AB + CD)^2 + (BC + AD)^2 = AB^2 + BC^2 + CD^2 + AD^2 + 2(AB \cdot CD + BC \cdot AD)$$

$$\geq AC^2 + BD^2 + 2 \cdot AC \cdot BD = (AC + BD)^2.$$

Equalities hold when $ABCD$ is a cyclic quadrilateral and a parallelogram. Therefore, in the equality case $ABCD$ is a rectangle.

Using the lemma, we have

$$(A_1A_2 + A_3A_4)^2 + (A_2A_3 + A_1A_4)^2 \geq (A_1A_3 + A_2A_4)^2.$$

Since $A_1A_2A_3A_4$ is a circumscribed quadrilateral with perimeter p_1 we get

$$p_1^2 = 2(A_1A_2 + A_3A_4)^2 + 2(A_2A_3 + A_1A_4)^2.$$

Since the sum of diagonal lengths is k_1, we get

$$p_1^2 \geq 2(A_1A_3 + A_2A_4)^2 = 2k_1^2.$$

Similarly, we obtain

$$p_2^2 \geq 2k_2^2.$$

Finally, by Cauchy-Schwarz inequality, we get

$$p_1^2 + p_2^2 \geq 2(k_1^2 + k_2^2) \geq (k_1 + k_2)^2.$$

The condition given in the problem corresponds to the equality case of the last inequality. By the equality case of lemma, each quadrilateral is a rectangle. Since they are circumscribed quadrilaterals, they should be squares. By the equality case of Cauchy-Schwarz inequality, we need $k_1 = k_2$ which shows that the side lengths of two squares are equal.