Bilkent University
Department of Mathematics

Problem Of The Month

April 2020

Problem:

Find all pairs of positive integers (a, b) satisfying the following equation:

$$\frac{a^3 + b^3}{ab + 4} = 2020.$$

Solution: Answer: $(1009, 1011)$ and $(1011, 1009)$.

Let us write the equation in the following form

$$(a + b)(a^2 - ab + b^2) = 4 \cdot 5 \cdot 101 \cdot (ab + 4) \qquad (1)$$

Let $p \in \{5, 101\}$. Then for some integer k we have $p = 6k + 5$. If $p|ab$ since $p|a^3 + b^3$ we get $p|a$ and $p|b$. Then in (1) the left hand is divisible by p^3 but the right hand side is not. Therefore, $p \not| ab$. Let b^{-1} be the inverse of b in $\pmod p$ and $c \equiv ab^{-1} \pmod p$. Since $p|a^3 + b^3$ we get $c^3 \equiv -1 \pmod p$. By Fermat's Theorem we have $c^{6k+4} \equiv 1 \pmod p$. Therefore, $c \equiv -1 \pmod p$ and consequently $p|a + b$. Thus, we get $505|a + b$. a and b are either both odd or both even.

Case 1: Both a and b are odd: By (1) we get $4|a + b$. Then $2020|a + b$ and $2020 \leq a + b$. From (1) $a^2 - ab + b^2 \leq ab + 4$. Therefore, $(a - b)^2 \leq 4$. $\Rightarrow |a - b| = 0$ or 2. Readily there is no solution for $a = b$. If $|a - b| = 2$ then $a + b = 2020$ and we get solutions $(1009, 1011)$ and $(1011, 1009)$.

Case 2: Both a and b even: Inserting $a = 2x$ and $b = 2y$ to (1) we get

$$(x + y)(x^2 - xy + y^2) = 2 \cdot 5 \cdot 101 \cdot (xy + 1) \qquad (2)$$

Either both x and y are even or both are odd. Therefore, $2|x+y$. Since $505|a+b = 2(x+y)$ we get $505|x + y$. Thus, $1010|x + y$ and $1010 \leq x + y$. By (2) we get $x^2 - xy + y^2 \leq xy + 1$. Therefore, $(x - y)^2 \leq 1$ and $|x - y| \leq 1$.Thus, $x = y$. Readily in this case (2) has no solution.