**Bilkent University**
**Department of Mathematics**

# Problem Of The Month

October 2019

**Problem:**

Let $x, y, z$ be real numbers satisfying $y > 2z > 4x$ and

$$2(x^3 + y^3 + z^3) + 15(xy^2 + yz^2 + zx^2) > 16(x^2y + y^2z + z^2x) + 2xyz.$$

Show that $4x + y > 4z$.

**Solution:**

Define $a = x - 2y, b = y - 2z, c = z - 2x$. Then $b, c > 0$ and the inequality $4x + y > 4z$ transfers to $b > 2c$. Now since

$$x = -\frac{a + 2b + 4c}{7},\ y = -\frac{b + 2c + 4a}{7}, z = -\frac{c + 2a + 4b}{7}$$

we get

$$\begin{aligned} S &= 16(x^2y + y^2z + z^2x) + 2xyz - 2(x^3 + y^3 + z^3) \\ &- 15(xy^2 + yz^2 + zx^2) = ab^2 + bc^2 + ca^2 - 2abc \\ &= c\left(a - b + \frac{b^2}{2c}\right)^2 + b(b - c)^2 + \frac{b^2(4c^2 - b^2)}{4c} < 0 \end{aligned}$$

Since $b, c > 0$, we get that $4c^2 < b^2$ and hence $b > 2c$.