Bilkent University
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Problem Of The Month

February 2019

Problem:

There are k heaps of beads containing 2019 beads in total. In each move we choose a heap: either remove it or divide it into two not necessarily equal parts. Find the maximal possible value of k such that for any initial distribution of beads after finite number of moves one can get k heaps with pairwise distinct number of beads.

Solution: Answer: 45.

If one can get k heaps with pairwise distinct number of beads, then the maximal heap among these contains at least k beads. Therefore, the maximal value of k can not be greater than 45. Indeed, if $k \geq 46$, then from k heaps each containing at most 45 beads one can not get k heaps with pairwise distinct number of beads but $45 \cdot k > 2019$.

Lemma. If k heaps contain at least $k(k-1)+1$ beads in total, then we can get k heaps containing $1, 2, \ldots, k$ beads.

Proof by induction on k. $k = 1$ is obvious. Suppose that the lemma is true for $k = n$. Consider $n+1$ heaps with $n(n+1)+1$ beads in total. Then the heap with maximal number of beads contains at least $n+1$ beads. If the heap with maximal number of beads contains exactly $n+1$ beads, the remaining n heaps contain $n(n+1)+1-(n+1) = n^2 \geq n(n-1)+1$ and by inductive hypothesis we can get n heaps containing $1, 2, \ldots, n$ beads. Since we also have a heap having $n+1$ beads we are done. If the heap with maximal number of beads contains more than $n+1$ beads, let us let us divide it into two heaps so that one of these two new heaps, say $H(n+1)$ contains $n+1$ beads. There are $n+1$ heaps except $H(n+1)$ containing $n(n+1)+1-(n+1) = n^2$ beads in total. The smallest heap among these $n+1$ heaps contains at most $n-1$ beads. If we remove it then the remaining n heaps contain at least $n^2-(n-1) = n(n-1)+1$ beads in total. By inductive hypothesis we can get n heaps containing $1, 2, \ldots, n$ beads. Since we also have a heap $H(n+1)$ we are done.

Since $2019 > 45 \cdot 44 + 1$ by lemma one can get heaps containing $1, 2, \ldots, 45$ beads. Done.