Bilkent University
Department of Mathematics

Problem Of The Month

January 2019

Problem:

Graphistan has 2019 cities and Graph Air (GA) is running one-way flights between all pairs of these cities. Determine the maximum possible value of the integer k such that no matter how these flights are arranged it is possible to travel from any city to any other city in Graphistan riding only GA flights so long as the absolute values of the difference between the number of flights originating and terminating at any city is not more than k.

Solution: Answer: 1009.

We want to find the largest integer k such that a directed path exists from any vertex v_1 to any vertex v_2 in any directed complete graph G with 2019 vertices satisfying $|\text{indeg}(v) - \text{outdeg}(v)| \leq k$ for all vertices v.

We first show that $k < 1010$ by giving a counterexample with $k = 2010$. Let us divide the vertices of G into two groups X and Y such that $|X| = 1009$ and $|Y| = 1010$. Let the vertices of X be placed around a circle and v be any vertex of X. Suppose that all edges between v and first 1004 vertices from v in clockwise direction are directed from v and all edges between v and first 1004 vertices from v in counter-clockwise direction are directed to v. Let us fix some vertex $v_0 \in Y$. Let us orient all vertices of complete graph $Y - v_0$ as in X. In Y there are 1009 edges incident to v_0. Let us direct 504 of these vertices from v_0 and 505 of these vertices to v_0. Finally, suppose that all edges between X and Y are directed to Y. It can be easily checked that in complete graph X $|\text{indeg}(v) - \text{outdeg}(v)| = 0$, in complete graph Y $|\text{indeg}(v) - \text{outdeg}(v)| \leq 1$ and in G $|\text{indeg}(v) - \text{outdeg}(v)| \leq 1010$ for any vertex v. Note that there is no directed path from any vertex of Y to any vertex of X.

Observe that $|\text{indeg}(v) - \text{outdeg}(v)|$ is always even. Now we will show that there is a directed path from any vertex to any vertex if $|\text{indeg}(v) - \text{outdeg}(v)| \leq 1008$ for all vertices v. Note that if one of $\text{indeg}(v)$ and $\text{outdeg}(v)$ is less than 505, then the other is greater than 1513 and $|\text{indeg}(v) - \text{outdeg}(v)|$ is at least 1010. Therefore $\text{indeg}(v)$ and $\text{outdeg}(v)$ are both at least 505. Let v_1 and v_2 be two distinct vertices in G. Let V_1 be the set of vertices that can be reached from v_1 and let V_2 be the set of vertices from which

v_2 can be reached. Then all arrows with tail in V_1 have also their heads in V_1. Therefore the sum of outdeg(v) for v in V_1 is $\binom{|V_1|}{2}$. On the other hand, this sum must be at least $505|V_1|$. It follows that $|V_1| \geq 1011$. A similar argument shows that $|V_2| \geq 1011$. Hence V_1 and V_2 are not disjoint. Done.