Bilkent University
Department of Mathematics

Problem Of The Month

September 2018

Problem:

Show that for each pair of positive integers (a, b) there is a positive integer n such that $n^2 + an + b$ has at least 2018 distinct prime divisors.

Solution:

By induction over k we will prove that for each pair of positive integers (a, b) there is a positive integer n such that $n^2 + an + b$ has at least k distinct prime divisors. The case $k = 1$ is obvious. For given (a, b) suppose that for some n_k $n_k^2 + an_k + b$ has k distinct prime divisors $p_1, p_2, \ldots, p_k$. Then $p_1 p_2 \ldots p_k$ divides $m = n_k^2 + an_k + b$. Let us define $n_{k+1} = n_k(m^2 + 1)$. Then

$$n_{k+1}^2 + an_{k+1} + b = n_k^2(m^2+1)^2 + an_k(m^2+1) + b$$

$$= n_k^2(m^4 + 2m^2 + 1) + an_k(m^2+1) + b = n_k^2 m^4 + 2m^2 n_k^2 + an_k m^2 + m$$

$$= m(n_k^2 m^3 + 2mn_k^2 + an_k m + 1).$$

The last expression is divisible by $p_1, p_2, \ldots, p_k$ and some another prime number p_{k+1} since m and $n_k^2 m^3 + 2mn_k^2 + an_k m + 1$ are relatively prime. The inductive hypothesis for $k + 1$ is proved. Done.