Bilkent University
Department of Mathematics

Problem Of The Month

July-August 2018

Problem:

Let x, y, z be positive real numbers such that

$$\sqrt{x}, \sqrt{y}, \sqrt{z} \text{ are sides of a triangle and } \frac{x}{y}+\frac{y}{z}+\frac{z}{x}=5.$$

Prove that

$$\frac{x(y^2-2z^2)}{z}+\frac{y(z^2-2x^2)}{x}+\frac{z(x^2-2y^2)}{y} \geq 0.$$

Solution:

Since $\sqrt{x}, \sqrt{y}, \sqrt{z}$ are sides of a triangle, we have

$$(\sqrt{x}+\sqrt{y}+\sqrt{z})(\sqrt{x}+\sqrt{y}-\sqrt{z})(\sqrt{y}+\sqrt{z}-\sqrt{x})(\sqrt{z}+\sqrt{x}-\sqrt{y})$$

$$= 2(xy+yz+zx)-x^2-y^2-z^2 \geq 0 \qquad (\dagger)$$

We also have

$$5xy = \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)xy = x^2+\frac{xy^2}{z}+yz$$
$$5yz = \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)yz = y^2+\frac{yz^2}{x}+zx$$
$$5zx = \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)zx = z^2+\frac{zx^2}{y}+xy$$

and by summing up the equations above, we get

$$2(xy+yz+zx)-(x^2+y^2+z^2) = \frac{xy^2}{z}+\frac{yz^2}{x}+\frac{zx^2}{y}-2(xy+yz+zx)$$

By the inequality ($\dagger$) the left hand side is non-negative. Therefore, the right hand side is also non-negative:

$$\frac{x(y^2-2z^2)}{z}+\frac{y(z^2-2x^2)}{x}+\frac{z(x^2-2y^2)}{y}\ge 0.$$

Remark. Assuming $x=\max\{x,y,z\}$, the equality holds when $\sqrt{\frac{y}{x}}=u$, $\sqrt{\frac{z}{x}}=1-u$ where $u\approx 0.555$ is the unique real root of $u^3-2u^2-u+1=0$ in the interval $(0,1)$.