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Problem Of The Month

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Problem:

Graph Air (GA) is running two way flights between some cities of a country so that it is possible to travel between any two cities using GA flights. For given cities A and B let $f(A, B)$ be the minimal number of flights necessary for travelling from A to B. The parameter of a company is $max f(A, B)$, where the maximum is taken over all possible pairs (A, B). It turned out that after adding one more flight to GA the parameter of GA became 33. Find the maximal possible value of the GA parameter before adding this flight.

Solution: The maximal value of GA parameter is 66.

Let $A_1, A_2, \ldots, A_{67}$ be cities and GA flights are set only between A_i and A_{i+1} for $i = 1, 2, \ldots, 66$. Then $f(A_1, A_{67}) = 66$. It can be easily seen that after adding flights between A_1 and A_{66}, the GA parameter becomes 33. Therefore, the answer is at least 66.

Let us show that 66 flights were sufficient for travelling between any two cities before adding the flight. On the contrary, suppose that $f(A, B) > 66$ for some cities A and B and the path with minimal number of flights between A and B is $(A = A_0, A_1, \ldots, A_{33}, A_{34}, \ldots, A_n = B)$. Suppose that the added flight is the flight between T and S. After adding this flight $f(A, A_{34}) \leq 33$ and $f(A_{33}, B) \leq 33$. By definitions, both paths with minimal number of flights from A to A_{34} and from A_{33} to B have to use the flight between T and S. Without loss of generality we may assume that the path from A to A_{34} is $(A_0, \ldots, T, S, \ldots A_{34})$. Then note that $l_1 + l_2 \leq 32$ where $f(A, T) = l_1, f(S, A_{34}) = l_2$. Similarly, the path from A_{33} to B is either $(A_{33}, \ldots, T, S, \ldots, B)$ with $f(A_{33}, T) = m_1, f(S, B) = m_2$ and $m_1 + m_2 \leq 32$, or $(A_{33}, \ldots, S, T, \ldots, B)$ with $f(A_{33}, S) = k_1, f(T, B) = k_2$ and $k_1 + k_2 \leq 32$. Then there exists a travel from A to B using less than 66 flights: it is either $(A, \ldots, T, \ldots, A_{34}, A_{33}, \ldots, S, \ldots, B)$ using at most $l_1 + m_1 + 1 + l_2 + m_2 < 66$ flights or $(A, \ldots, T, \ldots, B)$ using at most $l_1 + k_2 < 66$ flights. Contradiction with the assumption $f(A, B) > 66$ and we are done.