Bilkent University
Department of Mathematics

Problem Of The Month

Febuary 2018

Problem:

Find the largest real number T for which the inequality

$$\frac{x^2+1}{(x+y)^2+4(z+1)}+\frac{y^2+1}{(y+z)^2+4(x+1)}+\frac{z^2+1}{(z+x)^2+4(y+1)} \geq T$$

holds for all positive real numbers x, y and z.

Solution: Answer: $T = \frac{1}{2}$.

Note that $(x+y)^2 \leq 2(x^2+y^2)$ and $4z+4 \leq 2(z^2+3)$. Therefore,

$$\frac{x^2+1}{(x+y)^2+4(z+1)} \geq \frac{x^2+1}{2(x^2+y^2+z^2+3)}$$

Similarly we can write analogous inequalities for pairs (y, z) and (y, z). The sum of these three inequalities yields

$$\frac{x^2+1}{(x+y)^2+4(z+1)}+\frac{y^2+1}{(y+z)^2+4(x+1)}+\frac{z^2+1}{(z+x)^2+4(y+1)} \geq \frac{x^2+y^2+z^2+3}{2(x^2+y^2+z^2+3)} = \frac{1}{2}.$$

The left hand side equals $\frac{1}{2}$ at $x = y = z = 1$. Done.