Bilkent University
Department of Mathematics

Problem Of The Month

January 2018

Problem:

The sequence of positive integers $x_0, x_1, \ldots, x_{2018}$ is said to be a *new year* sequence if it satisfies the following three conditions:

† $\quad 1 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{2018}$

†† $\quad$ the range of the sequence consists of exactly 100 different positive integers

† † † $\quad \displaystyle\sum_{i=2}^{2018} x_i(x_i - x_{i-2}) = 9998.$

Find the number of distinct new year sequences.

Solution: Answer: $\displaystyle\binom{1918}{98} + \binom{98}{2}\binom{1918}{95}$.

Let us show that the number of sequences $1 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ consisting of k different positive integers and satisfying $\displaystyle\sum_{i=2}^{n} x_i(x_i - x_{i-2}) = k^2 - 2$ is equal to $\binom{n-k}{k-2} + \binom{k}{2}\binom{n-k}{k-5}$ for all $k \geq 5, n \geq 2k-2$. Note that due to monotonicity of the sequence for each $i \geq 2$

$$(x_i - x_{i-2} - 1)(x_i - x_{i-1}) \geq 0 \qquad (1)$$

Equivalently,

$$x_i^2 - x_i x_{i-2} + x_{i-1}x_{i-2} - x_i x_{i-1} \geq x_i - x_{i-1}$$

Side by side summation of (1) for $i = 2, 3, \ldots, n$ yields

$$\sum_{i=2}^{n} x_i(x_i - x_{i-2}) \geq x_n(x_{n-1} + 1) - 2x_1 \qquad (2)$$

Case 1: $x_1 = 1$. Since the sequence consists of k distinct integers we get

$$x_n \geq k \text{ and } x_{n-1} \geq k-1 \qquad (3)$$

Now by (2) we get

$$\sum_{i=2}^{n} x_i(x_i - x_{i-2}) \geq k^2 - 2$$

Therefore, the inequalities (1) and (3) should turn to equalities: $x_n = k, x_{n-1} = k-1$ and for $i = 2, 3, \ldots, n$ either $x_i - x_{i-2} = 1$ or $x_i = x_{i-1}$. Thus, $x_i - x_{i-1} \leq 1$. If for some i we have $x_i - x_{i-1} = x_{i-1} - x_{i-2} = 1$ then $x_i - x_{i-2} = 2$, a contradiction. It means that in the sequence the length of each block of coinciding elements is at least 2. In order to construct a sequence we should determine smallest indices for which the sequence element is equal to $2, 3, \ldots, k-1$. Evidently the number of choices is equal to the number of integer solutions of the equation $t_1 + t_2 + \cdots + t_{k-1} = n$, where each term is a positive integer exceeding 1. Thus the answer is $\binom{n-2(k-1)+k-1-1}{k-2} = \binom{n-k}{k-2}$.

Case 2: $x_1 \geq 2$. Since the sequence consists of k distinct integers we get

$$x_n \geq x_1 + k - 2 \text{ and } x_{n-1} \geq x_1 + k - 3 \qquad (4)$$

Now by (2) we get

$$\sum_{i=2}^{n} x_i(x_i - x_{i-2}) \geq (x_1 + k - 2)^2 - 2x_1$$

Therefore,

$$0 = \sum_{i=2}^{n} x_i(x_i - x_{i-2}) - (k^2 - 2) \geq (x_1 - 2)(x_1 + 2m - 4) - 2$$

If $x_1 > 2$ then the right hand side is positive, a contradiction. Therefore, $x_1 = 2$. Since $x_1 = 2$ we get $x_n \geq k, x_{n-1} \geq k-1$. If the least inequalities are not equalities then by (2) $\sum_{i=2}^{n} x_i(x_i - x_{i-2}) > k^2 - 2$. Therefore, $x_n = k$ and $x_{n-1} = k-1$. Thus, the new year sequence takes all values from 1 to k and $x_i - x_{i-1} \leq 1$. Therefore, the only possible non-zero value of the expression $(x_i - x_{i-2} - 1)(x_i - x_{i-1})$ in (1) is 1. Thus, the only possibility for $\sum_{i=2}^{n} x_i(x_i - x_{i-2}) = (k^2 - 2)$ is that for some two indices $(x_i - x_{i-2} - 1)(x_i - x_{i-1}) = 1$: $x_i - x_{i-1} = x_{i-1} - x_{i-2} = 1$ for two values of i. It means that in the sequence the length of each block of coinciding elements is at least 2 but there are two blocks with length 1. In order to construct a sequence we should determine the values of elements in blocks of length 1 and the smallest indices for which the sequence element is equal to $3, \ldots, k-1$. Evidently the number of choices is equal to the number of solutions of the equation $t_1 + t_2 + \cdots + t_{k-4} = n - 3$, where each term is a positive integer exceeding 1. Thus the answer is $\binom{k-2}{2}\binom{n-3-2(k-4)+k-4-1}{k-2} = \binom{k-2}{2}\binom{n-k}{k-5}$. Done.