Bilkent University
Department of Mathematics

Problem Of The Month

December 2017

Problem:

Show that for all positive real numbers $a_1, a_2, \ldots, a_{2017}$ satisfying $a_1 a_2 \cdots a_{2017} = 1$ the following inequality is held:

$$\sum_{i=1}^{2017} \frac{a_i}{1+a_i} \leq \frac{1}{2} \sum_{i=1}^{2017} \frac{1}{a_i}$$

Solution:

Let $f(a_1, a_2, \ldots, a_k) = \frac{1}{2} \sum_{i=1}^{k} \frac{1}{a_i} - \sum_{i=1}^{k} \frac{a_i}{1+a_i}$. We will prove by induction on k that

$f_k(a_1, a_2, \ldots, a_k) \geq 0$ for all positive real numbers $a_1, a_2, \ldots, a_k$ satisfying $a_1 a_2 \cdots a_k = 1$.

For $k = 1$ we get $a_1 = 1$ and $f_1(a_1) = 0$.

Suppose that $k > 1$ and $a_1 a_2 \cdots a_k = 1$. Without loss of generality suppose that $a_1 \leq a_2 \leq \cdots \leq a_k$. Let us show that

$$f(a_1, a_2, \ldots, a_k) - f_{k-1}(a_1 a_k, a_2, \ldots, a_{k-1}) = \frac{1}{2a_1} - \frac{a_1}{1+a_1} + \frac{1}{2a_k} - \frac{a_k}{1+a_k} - \left(\frac{1}{2a_1 a_k} - \frac{a_1 a_k}{1+a_1 a_k}\right) \geq 0 \qquad (1)$$

Indeed, after getting a common denominator the inequality (1) becomes

$$(1-a_1)(a_k-1)(2a_1^2 a_k^2 + a_1^2 a_k + a_1 a_k^2 + a_1 a_k + a_1 + a_k + 1) \geq 0$$

which is true since $0 < a_1 \leq 1 \leq a_k$. Thus,

$$f(a_1, a_2, \ldots, a_k) \geq f_{k-1}(a_1 a_k, a_2, \ldots, a_{k-1}) \geq 0$$

by inductive hypothesis. Done.