Bilkent University
Department of Mathematics

Problem Of The Month

July-August 2017

Problem:

There are 2017 nonempty bags each containing finite number of colored balls. Find the smallest value of k for which no matter how the contents of bags are arranged the bags can be distributed into k boxes so that for each box at least one of the following two conditions is held:

• all bags of a box contain a ball of the same color

•• each bag of a box contains a ball colored differently from all balls of all other bags of this box.

Solution: Answer: $k = 63$.

Let us show that $k \geq 63$. Suppose that there are 63 bags each containing one ball colored 1, 62 bags each containing one ball colored 2, ... , 1 bag containing one ball colored 63. Let us show that these 2016 bags can not be distributed into 62 boxes. Bag containing a ball colored l we call bag colored l. Note that if some box contains two bags colored l then all remaining bags are also colored l. A box containing only unicolored bags will be called a unicolored box. Consider any distribution of these 2016 bags into boxes. Suppose there are exactly s unicolored boxes. Then there is a color t so that there are at least $63 - s$ bags of color t. These bags should be in distinct boxes and the total number of boxes is at least $s + 63 - s = 63$. Done.

Now we prove that 63 boxes are sufficient for any arrangement of bag contents.

Lemma. Suppose that for some collection of bags the number of bags containing same colored ball is at most m. Then this collection can be distributed into at most m boxes.

Proof: Let us choose a color, say l_1. We can distribute all bags containing balls colored l_1 into distinct boxes, since the number of such bags is at most m. Among remaining bags choose a color, say l_2. Similarly we can distribute all bags containing balls colored l_2 into distinct boxes. Similarly, each time by choosing a new color will can distribute all bags into at most m boxes. Thus, each bag of any given box contains a ball colored differently from all balls of all other bags of this box. The lemma is proved.

At the first step consider a ball colored n_1 belonging to maximal number of bags. If this number is at most 63, by lemma we can distribute all bags into 63 boxes. If no, put all these bags (at least 64 bags) into one box and at the second step consider a ball colored n_2 belonging to maximal number of bags among all remaining bags. If this number is at most 62, by lemma we can distribute all bags into 62 boxes. If no, put all these bags (at least 63) having common ball numbered n_2 into one new box and proceed by the same way in the third step. Since $64+63+...+1 = 2080 > 2017$, at some step, say step number p among all remaining bags at most $64 - p$ bags will contain a ball of the same color n_p. Now by lemma we can distribute remaining bags into at most $64-p$ new boxes and since in the first $p-1$ steps we have used $p-1$ boxes at most 63 boxes will be sufficient. Done.