Bilkent University
Department of Mathematics

Problem Of The Month

May 2017

Problem:

Find all triples (m, n, p) satisfying

$$(m^3 + n)(n^3 + m) = p^3$$

where m, n are positive integers and p is a prime number.

Solution: Answer: $(m, n, p) = (2, 1, 3), (1, 2, 3)$.

If $m = n$ then $n^2(n^2 + 1) = p^3$. Since $gcd(n^2, n^2 + 1) = 1$ and p is prime we get $n = 1$, no solution.

Let $m > n \geq 1$. Since $n^3 + m \geq 2$ we get $m^3 + n = p^2(\dagger)$, $n^3 + m = p(\ddagger)$.

If $n = 1$ then $p^2 = (m^3 + 1) = (m + 1)(m^2 - m + 1)$ and $m + 1 = p$. Therefore, $m^2 - m + 1 = p = m + 1$, $m = 2$. Thus, we have a solution $m = 2$, $n = 1$, $p = 3$.

If $n > 1$ then $p = n^3 + m > m + n$ and therefore $p \not| m + n$ and $p \not| m - n$. Adding and subtracting of $(\dagger)$ and $(\ddagger)$ we get

$$(m + n)(m^2 - mn + n^2 + 1) = p(p + 1) \qquad (m - n)(m^2 + mn + n^2 - 1) = p(p - 1).$$

Since $p \not| m + n$ and $p \not| m - n$ we get $p \mid m^2 - mn + n^2 + 1$ and $p \mid m^2 + mn + n^2 - 1$. Therefore $p \mid (m^2 + mn + n^2 - 1) - (m^2 - mn + n^2 + 1) = 2(mn - 1)$. If $p = 2$ then $m = n = 1$, no solution. Therefore, $p \mid mn - 1$ and $p \leq mn - 1$. Now since $n^3 + n < n^3 + m = p \leq mn - 1 < mn$ we get $n^3 + n < mn$ and $n^2 + 1 < m$. Thus, $n^2 < m$. Since $p \leq mn - 1$ we get $p^2 \leq m^2n^2 - 2mn + 1$. Therefore, $m^3 + n = p^2 \leq m^2n^2 - 2mn + 1$. Since $n^2 < m$ we get $m^2n^2 - 2mn + 1 < m^3 - 2mn + 1$. Then $m^3 + n < m^3 - 2mn + 1$ and consequently $2mn + n < 1$, a contradiction.

The case $n > m \geq 1$ similarly yields the solution $n = 2$, $m = 1$, $p = 3$.