Bilkent University
Department of Mathematics

Problem Of The Month

March 2017

Problem:

Let $S_r(n) = 1^r + 2^r + \cdots + n^r$ where r is a rational number and n is a positive integer. Find all triples (a, b, c) where a and b are positive rational numbers and c is a positive integer for which there exist infinitely many positive integers n satisfying $S_a(n) = (S_b(n))^c$.

Solution: The answer: $a = 3, b = 1, c = 2$ and $a = b \in \mathbb{Q}^+, c = 1$.

By using of Bernoulli's inequality and induction on n we can easily get the inequality

$$\frac{n^{r+1}}{r+1} \leq S_r(n) \leq \frac{(n+1)^{r+1}}{r+1}$$

for all positive integer n and positive rational number r.

As $S_a(n) = (S_b(n))^c$ letting $r = a$ and $r = b$ we get

$$\frac{n^{a+1}}{a+1} \leq (\frac{(n+1)^{b+1}}{b+1})^c \text{ and } (\frac{n^{b+1}}{b+1})^c \leq \frac{(n+1)^{a+1}}{a+1}$$

Therefore, the inequalities

$$\frac{n^{(b+1)c}}{(n+1)^{a+1}} \leq \frac{(b+1)^c}{a+1} \leq \frac{(n+1)^{(b+1)c}}{n^{a+1}}$$

are held for infinitely many positive integers n. By letting $n \to \infty$ in the last inequality, we obtain that $(b+1)c = a+1$ and $(b+1)^c = a+1$. If $c = 1$, then $a = b$ and we get the trivial solutions.

If $c > 1$, then $c = (b+1)^{c-1}$ implies that b is an integer since c is an integer. As $b \geq 1$, we get that $c \geq 2^{c-1}$ and hence $c = 2$. This leads to $b = 1, a = 3$ and this solution clearly satisfies the conditions.